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Research paper

Equilibrium and stability of beams with elastoplastic semi-rigid joints

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Abstract: The classical theory of beams admits two types of connections between structural elements: perfectly rigid and perfectly flexible. However, the engineering reality calls for a more general approach to joint modelling. A non-ideal connection type is called semi-rigid. Its moment-rotation characteristic links the bending moment developed at the joint with the non-zero rotation increment between both sides of the connection. In the paper, the moment-rotation characteristic is assumed to be linearly elastic-perfectly plastic, and not sensitive to the sign of the rotation increment. Posed in the framework of Euler-Bernoulli's theory of beams, the main objective of the paper is to formulate slope-deflection equations for a beam with semi-rigid joints. Two themes are considered. The first regards the equations derived under the assumption of small transverse displacements; the results are not new, but they offer a different viewpoint. The variational form of the equilibrium problem involves minimization of stress energy functional with constraints following from the yield condition imposed on the values of moments in semi-rigid joints. Such an approach furnishes a theoretical foundation for robust numerical analysis of large-scale frames in the full spectrum of joint deformation: linearly elastic and perfectly plastic. The second theme of study concerns the beam subjected to a large compressive force. The results obtained on the grounds of Bleich's linear theory of stability include slope-deflection equations and the implicit definition of the critical value of compressive force. In both themes of interest, the outcomes of the paper essentially extend the findings documented in the literature.

Keywords: elastoplasticity, semi-rigid joints, slope-deflection equations, stability, variational formulation

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1. Introduction

The classical theory of beams admits only two types of connections: perfectly rigid and perfectly flexible. Based on statical and kinematical arguments, the classification of perfect joints is clear-cut. The rigid one transfers the bending moment, and the rotation angles of beam cross-sections at both sides of the joint are equal; put differently, the rotation increment at a rigid joint is zero. The mechanical characteristic of a flexible joint is opposite; it does not transfer the moment, and the rotation increment is arbitrary. In engineering reality, connections between structural elements do not always conform strictly to either category. Instead, they function somewhere between these two ideals. Such joints are considered *semi-rigid*; they transfer the bending moment and exhibit a non-zero rotation increment simultaneously.

Adopting semi-rigid joints in structural design requires a constitutive model, usually called the *moment-rotation characteristic*. A comprehensive discussion by Celik and Sakar [1] offers current state-of-the-art and recent developments regarding semi-rigid joints. Specifically, the authors recapitulated various models of the moment-rotation relation. Two aspects are of special interest for the present work. The first concerns the joints being stand-alone mechanical objects; in the same scope, Kozłowski and Bródka [2] compared and evaluated numerous semi-rigid constitutive models of joints in steel structures. The second aspect addresses the principles of the analytical and numerical approaches to frames with semi-rigid joints. Similar issues are also discussed in the review paper by Giżejowski et al. [3].

In this study, the moment-rotation characteristic of semi-rigid joints is set to be linearly elastic-perfectly plastic (elastoplastic, for short). The constitutive formula for semi-rigidity is symmetric concerning the sign of rotation increment; this assumption agrees with the conclusions of the research regarding progressive collapse in Kukla et al. [4]. The elastoplastic model is approved by design codes and standards (Eurocode [5], in particular) for the analysis of joints in steel structures subject to static loading.

The first objective is to formulate slope-deflection equations for a beam with semi-rigid joints under the assumption of small transverse displacements (deflections) within the classical Euler-Bernoulli beam theory. The problem is not new; it has been dealt with in many textbooks, e.g., Weaver and Gere [6] or Karnovsky and Lebed [7]. Results of the present study offer a different viewpoint based on the approach of Lewiński [8]. Slope-deflection equations in the new setting have a simple form despite involving more parameters than the equations used for calculations of beams with perfectly rigid joints.

The second objective is to obtain slope-deflection equations for the same beam subjected to a large compressive force. The study is based on the linear theory of stability by Bleich, cf. Życzkowski [9]. Partial results can be found in [7, 9], but detailed explanations are not documented in the literature. The implicit definition of the critical value of the compressive force is also a novel concept.

Discussion aligns with research concerning the optimum design of elastoplastic structures in Larsen et al. [10], who put forward a computational algorithm for identifying optimal beam grillage layouts, and the analysis by Czarnecki and Lewiński [11] concentrated on theoretical and numerical aspects of optimal design of a two-dimensional elastoplastic continuum.

Technical issues determining the moment-rotation characteristic of semi-rigid connections lie beyond the scope of the discussion in this paper. These are, for example, shapes of cross-sections of connected elements, technology used for the assembling of the connection, etc. Consequently, questions regarding experimental validation of theoretical formulas are not put forward. Interested readers are referred to [12] and other publications specialized in this topic.

The paper is organized as follows: in Section 2, slope-deflection equations for a slender beam with semi-rigid joints are derived, and the equilibrium problem in the variational form is formulated. Next, the discussion is reset to embrace the case of frames composed of an arbitrary number of slender elements interconnected by semi-rigid elastoplastic joints. In Section 3, the case of a beam subjected to a large compressive force is analyzed. Sections 2 and 3 contain examples illustrating the concepts. Discussion and conclusions are given in Section 4.

2. Slender beam with semi-rigid joints

2.1. Mechanical setting

Consider a straight, slender beam (Fig. 1) and assume that it is connected to adjacent nodes through joints that are semi-rigid concerning rotation about the y-axis (flexural rotation) and rigid otherwise.

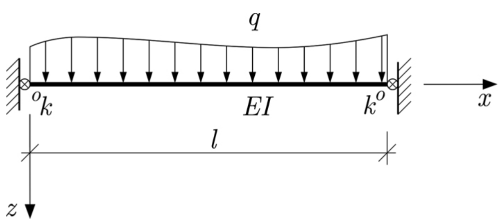

Fig. 1. Beam considered in Section 2. Symbol × at the beam ends represents a semi-rigid joint; other symbols are explained in the text. Axis y is perpendicular to the xz-plane and runs towards the observer

The beam is elastic, EI denotes its bending stiffness within the span $(0, l)$. We also set ${}^o k$ and k^o for the flexural stiffnesses of the left- and right-end joint, respectively. The values of ${}^o k$ and k^o, for which a joint may be classified as semi-rigid, are discussed in [5].

To measure transverse displacement, we introduce the deflection function w, and we write ${}^o w$, ${}^o\varphi$, and w^o, φ^o for deflections and angles of rotation of the left- and right-end node, respectively. In what follows, we use $f(g)$ to explicitly express the dependence of function f on another function g. Then, the value of $f(g)$ at given $x \in (0, l)$ will be denoted by $f(g)(x)$.

Applying the classical Euler-Bernoulli kinematical hypothesis for slender beams, we obtain

$$\varphi(w) = \frac{\mathrm{d}w}{\mathrm{d}x}, \quad \kappa(w) = -\frac{\mathrm{d}\varphi(w)}{\mathrm{d}x} \tag{2.1}$$

where: φ – angle of rotation of the cross-section, κ – bending deformation measure.

Jump discontinuity of φ is allowed at semi-rigid joints. This yields the finite increment of angle of rotation, or the *rotation increment* for short. It is defined by

$$\eta(w) = -\left(\varphi^{+}(w) - \varphi^{-}(w)\right) \tag{2.2}$$

where: φ^{+}, φ^{-} – angles of rotation on both sides of the semi-rigid joint, respectively.

Since joints are situated at beam ends, then (Fig. 2a),

$$\eta(w)(0) = -\left(\frac{\mathrm{d}w}{\mathrm{d}x}(0) - {}^{o}\varphi\right), \quad \eta(w)(l) = -\left(\varphi^{o} - \frac{\mathrm{d}w}{\mathrm{d}x}(l)\right) \tag{2.3}$$

but a more convenient notation is ${}^{o}\eta(w) = \eta(w)(0)$ and $\eta^{o}(w) = -\eta(w)(l)$,

$${}^{o}\eta(w) = {}^{o}\varphi - \frac{\mathrm{d}w}{\mathrm{d}x}(0), \quad \eta^{o}(w) = \varphi^{o} - \frac{\mathrm{d}w}{\mathrm{d}x}(l) \tag{2.4}$$

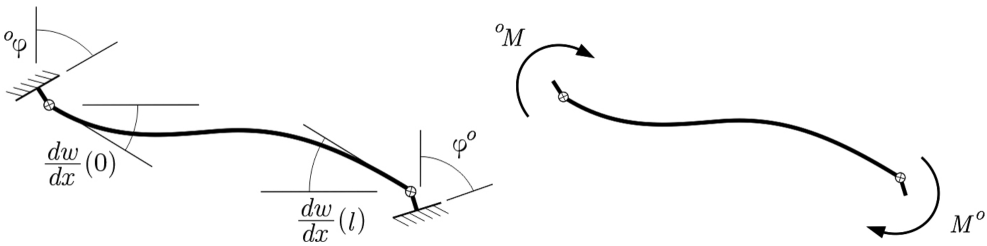

Fig. 2. Graphical explanation of the relations in Eq. (2.3)–(2.6)

We call a function w *kinematically admissible*, and denote it $w \in \mathcal{K}$, if the following two requirements are met:

i) w is continuous in the span of the beam and satisfies the boundary conditions

$$w(0) = {}^{o}w, \quad w(l) = w^{o} \tag{2.5}$$

ii) φ is continuous in the span of the beam and admits rotation increments at semi-rigid joints.

Let M stand for the bending moment and denote the values of M at the left- and right-end of the beam, respectively, as (Fig. 2b),

$${}^{o}M = M(0), \quad M^{o} = -M(l) \tag{2.6}$$

Note that left- and right-end bending moments defined in this way are co-directed with the corresponding rotation increments in Eq. (2.4).

Using standard arguments of the classical theory of beams, we get the variational equation of beam equilibrium,

$$\int_{0}^{l} M\kappa(w)\,\mathrm{d}x + {}^{o}M{}^{o}\eta(\overline{w}) + M^{o}\eta^{o}(\overline{w}) = f(\overline{w}) \ \ \forall\, \overline{w} \in \mathcal{K} \tag{2.7}$$

where:

$$f(\overline{w}) = \int_0^l q\overline{w}\mathrm{d}x + M^o\chi^o + {}^oM\,{}^o\chi \tag{2.8}$$

and

$${}^o\chi = {}^o\varphi - \frac{w^o - {}^ow}{l}, \quad \chi^o = \varphi^o - \frac{w^o - {}^ow}{l} \tag{2.9}$$

The moment-rotation (or rather, moment-rotation increment) characteristic of a semi-rigid joint is assumed to be elastoplastic; hence, ${}^oM$ and M^o are constrained by yield conditions

$$|{}^oM| \le {}^oM_{\mathrm{pl}}, \quad |M^o| \le M^o_{\mathrm{pl}} \tag{2.10}$$

where: ${}^oM_{\mathrm{pl}}, M^o_{\mathrm{pl}}$ – plastic limits of moments developed in the left- and right-end semi-rigid joints, respectively.

We will write $M \in \mathcal{S}$ to denote that M is in equilibrium with q in the sense of Eq. (2.7). Moreover, we set $M \in \mathcal{H}$ for M satisfying the yield conditions in Eq. (2.10). Function M is said to be *statically admissible* if $M \in \mathcal{S} \cap \mathcal{H}$.

Two possible scenarios of mechanical response of the beam follow from the conditions in Eq. (2.10). In the first scenario, the values of ${}^oM$ and M^o are strictly lower than their respective thresholds, so relations in Eq. (2.10) are realized as strict inequalities. Then, the span and both end joints work in the elastic range, and there exist $w \in \mathcal{K}$ and $M \in \mathcal{S} \cap \mathcal{H}$ corresponding to the equilibrium of the beam. They are *compatible* in the sense that they are linked by linear relations,

$$\kappa(w) = \frac{M}{EI}, \quad {}^o\eta(w) = \frac{{}^oM}{{}^ok}, \quad \eta^o(w) = \frac{M^o}{k^o} \tag{2.11}$$

In the second scenario, at least one relation in Eq. (2.10) is realized as equality. The beam falls into the elastoplastic range; functions w and M still correspond to the equilibrium of the beam, but their compatibility is retained only in the span $(0, l)$, which means that only the first identity in Eq. (2.11) remains valid. The second and third are subject to modifications put forward in Section 2.3.

2.2. Slope-deflection equations

The task is now to discuss in greater detail the bending problem posed for the beam in the elastic work regime. Our specific goal is to derive *slope-deflection equations*, which are the relations linking the left- and right-end moments with the left- and right-end deflections and rotations. We emphasize that slope-deflection equations are insensitive to the load q applied within span $(0, l)$. Hence, we temporarily assume $q = 0$. We also set $EI(x) = \mathrm{const}$.

The reasoning leading to the definition of w is based on standard techniques of the mechanics of structures. As a result, we get the differential equation,

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}(x) = 0 \tag{2.12}$$

with boundary conditions

$$w(0) = {}^{o}w, \quad M(0) = {}^{o}t\frac{EI}{l}\left({}^{o}\varphi - \frac{\mathrm{d}w}{\mathrm{d}x}(0)\right)$$
$$w(l) = w^{o}, \quad -M(l) = t^{o}\frac{EI}{l}\left(\varphi^{o} - \frac{\mathrm{d}w}{\mathrm{d}x}(l)\right) \tag{2.13}$$

where: ${}^{o}t = {}^{o}kl/(EI)$ and $t^{o} = k^{o}l/(EI)$.

Tedious but straightforward algebraic manipulations finally give slope-deflection equations for left- and right-end moments. Equations (2.6) become now

$$ {}^{o}M = \frac{EI}{l}\left(4F_1\left({}^{o}t, t^{o}\right){}^{o}\chi + 2F_2\left({}^{o}t, t^{o}\right)\chi^{o}\right)$$
$$M^{o} = \frac{EI}{l}\left(2F_2\left(t^{o}, {}^{o}t\right){}^{o}\chi + 4F_1\left(t^{o}, {}^{o}t\right)\chi^{o}\right) \tag{2.14}$$

where:

$$F_1(a, b) = \frac{a(b+3)}{ab + 4(a+b) + 12}$$
$$F_2(a, b) = \frac{ab}{ab + 4(a+b) + 12} \tag{2.15}$$

We leave it to the reader to verify that passing with arguments of functions in Eq. (2.15) to their extremes (0 or $+\infty$) leads to slope-deflection equations for beams with ideal joints. For the reason of completeness of the discussion, let us mention that in the case $q \neq 0$, the span load induces additional end moments, which must be added to the right-hand sides of formulae in Eq. (2.14).

Deflection functions are not reported explicitly due to their complexity; a simple example is provided in Section 2.3.

Consider now the energy functional

$$\Phi(M) = \int_0^l \frac{M^2}{EI}\mathrm{d}x + \frac{({}^{o}M)^2}{{}^{o}k} + \frac{(M^{o})^2}{k^{o}} \tag{2.16}$$

According to Duvaut and Lions [13], the variational problem

$$(P):\ \min\left\{\Phi\left(\overline{M}\right) \mid \overline{M} \in \mathcal{S} \cap \mathcal{H}\right\} \tag{2.17}$$

furnishes the bending moment function corresponding to the equilibrium of the beam. Note that (P) embraces the full spectrum of mechanical response, including both elastic and elastoplastic regimes of semi-rigid joints.

When plastic deformations, denoted henceforth by ${}^{o}\eta_{\mathrm{pl}}$ and η^{o}_{pl}, start to develop in semi-rigid joints, then the second and third relations in Eq. (2.11) become

$$ {}^{o}\eta(w) = \frac{{}^{o}M}{{}^{o}k} + {}^{o}\eta_{\mathrm{pl}}, \quad \eta^{o}(w) = \frac{M^{o}}{k^{o}} + \eta^{o}_{\mathrm{pl}} \tag{2.18}$$

We emphasize that ${}^o\eta_{\mathrm{pl}}$ and η^o_{pl} are not compatible with any kinematically admissible deflection function. Nevertheless, the complete solution to the equilibrium problem in the elastoplastic regime of deformation is possible. Indeed, using the methods of Lagrange duality, cf. Luenberger [14], one may derive a dual variational problem, say (P^*). It turns out to be the maximum problem posed in terms of variable $w \in \mathcal{K}$. Functions $M \in \mathcal{S} \cap \mathcal{H}$ and $w \in \mathcal{K}$ solving the problems (P) and (P^*), respectively, are obtained independently. In the last step, one may apply the solutions in the first identity in Eq. (2.11), and the relations in Eq. (2.18).

2.3. Example

For a simple example illustrating the discussion above, consider the beam in Fig. 3. The left-end joint of the beam is semi-rigid, and the right-end one is perfectly rigid. The beam is subjected to deflection w^o of the right-end node. Our interest is in calculating the limit value of w^o, for which the moment in the left-end joint achieves the threshold $|{}^oM| = {}^oM_{\mathrm{pl}}$.

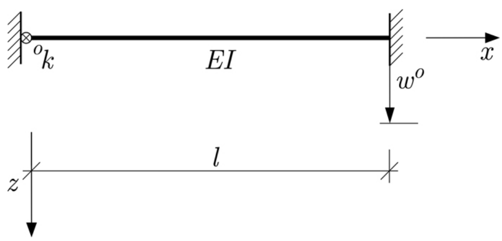

Fig. 3. Beam from Example 2.4

We solve Eq. (2.12), applying ${}^ow = {}^o\varphi = \varphi^o = 0$ in the boundary conditions Eq. (2.13). As a result, we get

$$ {}^oM = -\frac{6EI}{l}\,\frac{{}^ot}{{}^ot+4}\,\frac{w^o}{l} \tag{2.19}$$

Then, from $|{}^oM| = {}^oM_{\mathrm{pl}}$, it follows that

$$ w^o_{\mathrm{limit}} = -\frac{l^2}{6EI}\,\frac{{}^ot+4}{{}^ot}\,{}^oM_{\mathrm{pl}} \tag{2.20}$$

2.4. Extension to frames with semi-rigid joints

Let us now extend the discussion above to the case of a frame composed of m members (bars) with semi-rigid joints; the neutral axes of bars intersect at n nodes. The frame is assumed to be kinematically stable under the action of any load applied to it. Proper support constrains the displacements of nodes. Set d for the number of unconstrained nodal displacements, typically referred to as degrees of freedom, and introduce $\mathbf{v} \in R^d$ to denote the vector of degrees of freedom. Any such vector is kinematically admissible; by abuse of notation, we write $\mathbf{v} \in \mathcal{K}$.

Each member of the frame undergoes elastic deformation of two types: axial and flexural. The latter is limited to the local xz-plane, but extending the results to the case of general bending and torsion (without warping) is immediate and thus omitted here for clarity. For the treatment of beam deformation, we introduce vectors:

$$\boldsymbol{\Delta}(\mathbf{v}) = \mathbf{B}\mathbf{v}, \quad \boldsymbol{\chi}(\mathbf{v}) = {}^{o}\mathbf{B}\mathbf{v}, \quad \boldsymbol{\chi}^{o}(\mathbf{v}) = \mathbf{B}^{o}\mathbf{v} \tag{2.21}$$

where: $\mathbf{B}$ – axial deformation matrix, ${}^{o}\mathbf{B}$, $\mathbf{B}^{o}$ – bending deformation matrices. The matrices are elements of $R^{m\times d}$ constructed according to the well-known principles of finite element programming.

With $e = 1, \ldots, m$ labeling the components of deformation vectors, we have

$$\Delta_e = u_e^o - {}^{o}u_e \tag{2.22}$$

and the definitions of ${}^{o}\chi_e(\mathbf{v})$ and $\chi_e^o(\mathbf{v})$ follow from Eq. (2.9). Symbols ${}^{o}u_e$, ${}^{o}w_e$, ${}^{o}\varphi_e$, and u_e^o, w_e^o, φ_e^o, denote axial displacements, transverse displacements (deflections), and angles of rotation of the left-end node and right-end node of the e-th element, respectively.

Next, we consider square diagonal constitutive matrices (elements of $R^{m\times m}$),

$$\mathbf{E} = \operatorname{diag}\left(\frac{EA_1}{l_1}, \ldots, \frac{EA_m}{l_m}\right) \tag{2.23}$$

where: EA_e – axial stiffness, A_e – cross-section area of e-th element and

$$\mathbf{D}' = \operatorname{diag}\left(D_1', \ldots, D_m'\right), \quad \mathbf{D}'' = \operatorname{diag}\left(D_1'', \ldots, D_m''\right) \tag{2.24}$$

with, cf. Eq. (2.15),

$$D_e' = \frac{EI_e}{l_e} F_1\left({}^{o}t_e, t_e^o\right), \quad D_e'' = \frac{EI_e}{l_e} F_2\left({}^{o}t_e, t_e^o\right) \tag{2.25}$$

Let $\mathbf{N}$, ${}^{o}\mathbf{M}$, and $\mathbf{M}^{o}$ denote the vectors (elements of R^{m}) of axial forces and left- and right-end moments respectively. They are given by

$$\mathbf{N} = \mathbf{E}(\mathbf{B}\mathbf{v}) \tag{2.26}$$

$${}^{o}\mathbf{M} = 4\mathbf{D}'({}^{o}\mathbf{B}\mathbf{v}) + 2\mathbf{D}''(\mathbf{B}^{o}\mathbf{v}), \quad \mathbf{M}^{o} = 2\mathbf{D}'({}^{o}\mathbf{B}\mathbf{v}) + 4\mathbf{D}''(\mathbf{B}^{o}\mathbf{v}) \tag{2.27}$$

Following Eq. (2.10), we set the yield conditions

$$|{}^{o}\mathbf{M}| \le {}^{o}\mathbf{M}_{\mathrm{pl}}, \quad |\mathbf{M}^{o}| \le \mathbf{M}_{\mathrm{pl}}^{o} \tag{2.28}$$

where: ${}^{o}\mathbf{M}_{\mathrm{pl}}$, $\mathbf{M}_{\mathrm{pl}}^{o}$ – vectors of plastic limits of bending moments developed in semi-rigid joints. Both '$\le$' relations must be satisfied elementwise.

We leave it to the reader to verify that the equilibrium equation for all nodes of the frame states

$$\mathbf{B}^{\mathrm{T}}\mathbf{N} + ({}^{o}\mathbf{B})^{\mathrm{T}}\,{}^{o}\mathbf{M} + (\mathbf{B}^{o})^{\mathrm{T}}\,\mathbf{M}^{o} = \mathbf{P} \tag{2.29}$$

where: $\mathbf{P}$ – vector of nodal loads (element of R^{d}).

Definition of statical admissibility introduced in Section 2.1 for a single bar naturally extends to bar structures. We write $(\mathbf{N}, {}^{o}\mathbf{M}, \mathbf{M}^{o}) \in \mathcal{S}$ to denote that $\mathbf{N}$, ${}^{o}\mathbf{M}$, and $\mathbf{M}^{o}$ are in equilibrium in the sense of Eq. (2.29). Moreover, we set $({}^{o}\mathbf{M}, \mathbf{M}^{o}) \in \mathcal{H}$ for ${}^{o}\mathbf{M}$ and $\mathbf{M}^{o}$ satisfying the yield conditions in Eq. (2.28). A triplet $(\mathbf{N}, {}^{o}\mathbf{M}, \mathbf{M}^{o}) \in \mathcal{S} \cap \mathcal{H}$ is said to be *statically admissible*.

If semi-rigid joints operate in the elastoplastic mode of deformation, then $(\mathbf{N}, {}^{o}\mathbf{M}, \mathbf{M}^{o})$ corresponding to the equilibrium of the frame is obtained via the minimum stress energy problem, cf. the discussion in Section 2.2. By abuse of notation, define the stress energy functional for the frame

$$\Phi\left(\mathbf{N}, {}^{o}\mathbf{M}, \mathbf{M}^{o}\right) = \sum_{e=1}^{m} \left(\frac{(N_e)^2}{EA_e} l_e + \frac{Q\left({}^{o}M_e, M_e^{o}\right)}{EI_e} \right) + \sum_{e=1}^{m} \left(\frac{({}^{o}M_e)^2}{{}^{o}k_e} + \frac{\left(M_e^{o}\right)^2}{k_e^{o}} \right) \tag{2.30}$$

where: $Q({}^{o}M_e, M_e^{o})$ – the integral of $(M_e)^2$ over the span (l_e) in case of linear distribution of bending moment function M_e.

The equilibrium problem

$$(P_{\#}): \quad \begin{array}{ll} \text{Minimize} & \Phi\left(\mathbf{N}, {}^{o}\mathbf{M}, \mathbf{M}^{o}\right) \\ \text{subject to:} & \mathbf{B}^{\mathrm{T}}\,\mathbf{N} + ({}^{o}\mathbf{B})^{\mathrm{T}}\,{}^{o}\mathbf{M} + (\mathbf{B}^{o})^{\mathrm{T}}\,\mathbf{M}^{o} = \mathbf{P} \\ & |{}^{o}\mathbf{M}| \le {}^{o}\mathbf{M}_{\mathrm{pl}} \\ & |\mathbf{M}^{o}| \le \mathbf{M}_{\mathrm{pl}}^{o} \end{array} \tag{2.31}$$

embraces the full spectrum of mechanical response of the frame, including both elastic and elastoplastic regimes of semi-rigid joints. In this meaning, $(P_{\#})$ is an extension of (P) put forward in Section 2.2.

3. Stability

3.1. Slope-deflection equations

Consider now the problem of a slender, straight, prismatic, and homogeneous beam with semi-rigid joints at both ends, subjected to a large axial compressive force S. Here, we follow Bleich's theory, cf. Życzkowski [9], assuming that the value of $S > 0$ is fixed.

In this scope, the discussion is limited to deriving the formula for the critical value S_{cr} of the large compressive force under classical assumptions of Euler's theory of stability [9]. In this context, the straight-line equilibrium configuration of the beam becomes unstable for $S = S_{\mathrm{cr}}$, and the possible failure is due to purely elastic buckling. The axial stress corresponding to S_{cr} does not exceed the yield stress for the material of the beam; this requirement is satisfied for slender bars made of materials typically used for civil engineering constructions. Post-critical behavior, i.e., analysis of the equilibrium path for $S > S_{\mathrm{cr}}$ is not tackled in this study.

Following Bleich's approach, large axial force $\mathcal{S}$ is accounted for in the equilibrium equation of the beam with respect to bending. With this, the variational formula in Eq. (2.7) becomes,

$$\int_0^l \left(M\kappa\left(\overline{w}\right) - S\varphi\left(w\right)\varphi\left(\overline{w}\right)\right)\mathrm{d}x + {}^{o}M^{o}\eta\left(\overline{w}\right) + M^{o}\eta^{o}\left(\overline{w}\right) = f\left(\overline{w}\right)\ \forall\,\overline{w} \in \mathcal{K} \tag{3.1}$$

where: $f(\overline{w})$ – as in Eq. (2.8), $w \in \mathcal{K}$ – the actual deflection function.

The differential formula in Eq. (2.12) is replaced by

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}(x) + \frac{S}{EI}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(x) = 0 \tag{3.2}$$

with boundary conditions as in Eq. (2.13). Assuming temporarily that ${}^{o}t = t^{o} = t$, and setting the axial load parameter σ, such that

$$\sigma^2 = \frac{Sl^2}{EI} \tag{3.3}$$

we get the counterparts of formulae in Eq. (2.14). They are given by

$$\begin{aligned} {}^{o}M &= \frac{EI}{l}\left(G_1(t,\sigma)\,{}^{o}\chi + G_2(t,\sigma)\,\chi^{o}\right) \\ M^{o} &= \frac{EI}{l}\left(G_2(t,\sigma)\,{}^{o}\chi + G_1(t,\sigma)\,\chi^{o}\right) \end{aligned} \tag{3.4}$$

where:

$$\begin{aligned} G_1(a,\theta) &= \frac{a^2\theta(\sin\theta - \theta\cos\theta) + a\theta^3\sin\theta}{\theta^3\sin\theta + 2a\theta(\sin\theta - \theta\cos\theta) + a^2(2 - 2\cos\theta - \theta\sin\theta)} \\ G_2(a,\theta) &= \frac{a^2\theta(\theta - \sin\theta)}{\theta^3\sin\theta + 2a\theta(\sin\theta - \theta\cos\theta) + a^2(2 - 2\cos\theta - \theta\sin\theta)} \end{aligned} \tag{3.5}$$

3.2. Critical values of axial forces

Considering Eq. (3.2) with ${}^{o}w = w^{o} = {}^{o}\varphi = \varphi^{o} = 0$ in the boundary conditions Eq. (2.13), we turn to the Euler's critical load problem. Setting back the different values of ${}^{o}t$ and t^{o}, we obtain the implicit definition of σ_{cr},

$$d\left({}^{o}t, t^{o}, \sigma_{\mathrm{cr}}\right) = 0 \tag{3.6}$$

where: σ_{cr} – critical value of axial load parameter σ, and

$$d(a,b,\theta) = \theta^3\sin\theta + \theta(\sin\theta - \theta\cos\theta)(a+b) + ab(2 - 2\cos\theta - \theta\sin\theta) \tag{3.7}$$

Note that the denominators in Eq. (3.5) are given by $d(a,a,\theta)$.

3.3. Example

Figure 4 displays the contour plot of function $d\left({}^{o}t, t^{o}, \sigma_{\mathrm{cr}}\right) = 0$. Each contour corresponds to some value of σ_{cr} in the interval $(\pi, 2\pi)$. The critical load for a beam with perfectly flexible joints (pins) at both ends is defined through Eq. (3.3) for $\sigma_{\mathrm{cr}} = \pi$, and $\sigma_{\mathrm{cr}} = 2\pi$ gives S_{cr} for a beam whose connections at the ends are perfectly rigid. These two solutions are unique in the sense that they relate to single points: ${}^{o}t = t^{o} = 0$ and ${}^{o}t = t^{o} = +\infty$, respectively.

Intermediate values of σ_{cr} realize for infinitely many pairs $({}^{o}t,\ t^{o})$. For example, the critical value $\sigma_{\mathrm{cr}} \approx 1.43\pi$ pertains not only to a beam with perfectly rigid and perfectly flexible supports at opposite ends (${}^{o}t = 0$ and $t^{o} = +\infty$), but also for a variety of semi-rigid connections. In particular, $\sigma_{\mathrm{cr}} \approx 1.43\pi$ defines the critical load also in case of ${}^{o}t = t^{o} = 3.59$.

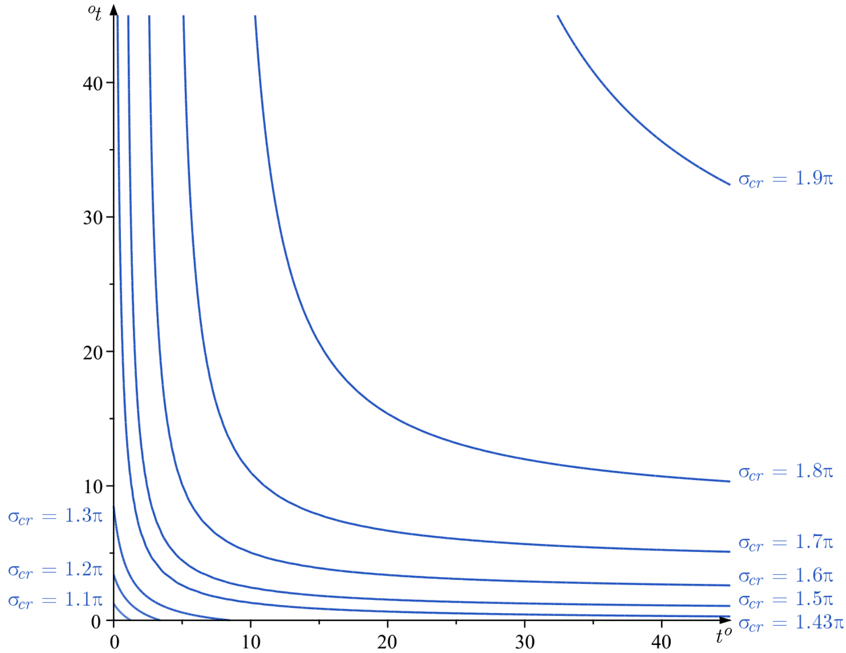

Fig. 4. Contour plot of the function defined by Eq. (3.6). The curve $\sigma_{\mathrm{cr}} = 1.43\pi$ is discussed in the text

4. Conclusions

The basic problem of a single beam with semi-rigid elastoplastic joints furnishes a theoretical foundation for robust numerical analysis of large-scale frames with semi-rigid connections in the full spectrum of joint deformation: linearly elastic and perfectly plastic. This is due to the computational possibilities offered by the variational formulation in Eq. (2.17) and Eq. (2.30). Large-scale bar constructions require algebraic formulation of equilibrium equations, but this should pose no additional problem, as appropriate procedures are well known in finite element programming.

The minimum stress energy problem falls into the framework of convex optimization. Full discussion of this issue requires specialized mathematical tools and thus lies beyond the scope of the present study. Loosely speaking, the main argument lies in the fact that the stress energy functional and the constraints define convex sets.

Another possible application of the principles governing the large-scale problem is the optimum design of frames with semi-rigid joints. Csébfalvi [15] tackled this topic in the elastic range of joint behavior by means of the genetic algorithm. Generalizing the approach to embrace the elastoplastic range seems to be an open problem.

In the scope of stability, it is interesting to construct the so-called *buckling stability region*. By this, we mean considering two or more large compressive forces acting simultaneously along the axes of several elements of the frame and determining the range of values for which the elements do not buckle.

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Równowaga i stateczność prętów ze sprężystoplastycznymi węzłami podatnymi

Słowa kluczowe: sformułowanie wariacyjne, sprężystoplastyczność, stateczność, węzły podatne, wzory transformacyjne

Streszczenie:

Klasyczna teoria Eulera–Bernoulliego prętów zginanych uwzględnia jedynie dwa rodzaje połączeń: idealnie sztywne i idealnie podatne (przegubowe). Różnicę między nimi łatwo określić na podstawie rozważań statycznych i kinematycznych. Węzły podatne przekazują moment zginający, zaś kąty obrotu przekrojów po obu stronach połączenia są jednakowe; inaczej mówiąc, przyrost kąta obrotu jest równy zeru. Charakterystyka przegubu jest skrajnie przeciwna. Moment nie jest przekazywany, a przyrost kąta obrotu jest dowolny. Praktyka inżynierska pokazuje jednak, że rzeczywiste połączenia elementów konstrukcji wymykają się tej klasyfikacji. Obok połączeń idealnych istnieją węzły podatne, nazywane także półsztywnymi. Są one zdolne do przekazywania momentu zginającego przy różnym od zera przyroście kąta obrotu przekroju. Włączenie węzłów podatnych do katalogu rozwiązań projektowych wymaga opracowania konstytutywnego modelu połączenia, zwykle nazywanego krzywą momentu-obrotu.

Autorzy publikacji [1] wyczerpująco omawiają stan współczesnej wiedzy i kierunki rozwoju teorii konstrukcji stalowych z węzłami podatnymi. Wśród wielu podjętych w [1] wątków, dwa są szczególnie interesujące w odniesieniu do zagadnień poruszanych w niniejszej pracy. Pierwszy dotyczy rozmaitych propozycji przebiegu krzywej momentu-obrotu; ten temat jest omawiany także w obszernej monografii [2]. Drugi wątek obejmuje analityczne i numeryczne rozważania na temat zasad projektowania ram z węzłami podatnymi. Podobne zagadnienie jest treścią przeglądowej publikacji [3]. Przedłożona praca opiera się na założeniu funkcji momentu-obrotu opisującej sprężystoplastyczne (ściślej liniowo sprężyste-idealnie plastyczne) prawo konstytutywne węzła podatnego. Związek fizyczny jest symetryczny względem znaku przyrostu kąta obrotu. Przyjęcie takiej formuły wpisuje się w wyniki badań dotyczących zagadnienia postępującej katastrofy, por. [4]. W świetle norm projektowych, np. Eurokodu [5], stosowanie modelu konstytutywnego sprężystoplastyczności w opisie zachowania węzła podatnego jest dopuszczalne w analizie połączeń elementów konstrukcji stalowych poddanych obciążeniom statycznym. Głównym celem pracy było opisanie zadania równowagi pręta zginanego z węzłami sprężystoplastycznymi umieszczonymi na obu krańcach. W szczególności podjęto dwa zadania. Pierwsze dotyczyło wyprowadzenia wzorów transformacyjnych w ramach klasycznej teorii Eulera–Bernoulliego, zaś drugie obejmowało przypadek dodatkowego obciążenia osiową siłą ściskającą o znacznej, ale znanej wartości, zgodnie z wymaganiami liniowej teorii Bleicha, por. [9]. W tym zakresie rozważań wyprowadzono także niejawną formułę określającą wartość siły krytycznej w zagadnieniu wyboczenia pręta smukłego. Wyniki uzyskane w zakresie teorii klasycznej pokrywają się z wynikami znanymi z literatury, jednak ponowne ich wyprowadzenie w duchu pracy [8] pozwoliło na istotne uproszczenie zapisu. Ma to szczególne znaczenie wobec zwiększonej liczby parametrów zadania zginania pręta z węzłami podatnymi w porównaniu z zadaniem, w którym węzły są idealnie sztywne. Badania w zakresie uwzględniającym działanie dużej siły osiowej były prowadzone przez autorów monografii [7, 9], ale podane rezultaty są niepełne. Literatura nie podaje także formuły pozwalającej określić wartość siły krytycznej w zagadnieniu wyboczenia pręta z węzłami podatnymi, co skłania do wniosku, że związki wyprowadzone w niniejszej pracy były dotychczas nieznane. Cel rozważań jest zbieżny z badaniami prowadzonymi w zakresie optymalnego projektowania konstrukcji, por. [10], gdzie przedstawiono wyniki dotyczące rusztów sprężystoplastycznych, a także [11], gdzie omówiono zadanie optymalizacji sprężystoplastycznych płyt obciążonych w swojej płaszczyźnie. Układ pracy jest następujący. W rozdziale 2 wyprowadzono wzory transformacyjne zginania pręta z węzłami podatnymi rozmieszczonymi na obu krańcach oraz sformułowano wariacyjne zagadnienie równowagi polegające na minimalizacji energii komplementarnej z ograniczeniami wartości momentów w węzłach podatnych zgodnie z przyjętym warunkiem uplastycznienia. Następnie, rozszerzono rozważania na przypadek ram z dowolną liczbą prętów. W rozdziale 3 omówiono przypadek pręta poddanego działaniu dużej siły ściskającej. Treścią rozdziału 4 są wnioski z pracy. Główne spostrzeżenie dotyczy możliwości, jakie niesie ze sobą wariacyjne sformułowanie zadania minimum energii komplementarnej. Rozszerzenie tego ujęcia na przypadek konstrukcji wieloprętowych z węzłami pracującymi w pełnym zakresie sprężystoplastyczności wymaga zastosowania metod algebraizacji równań równowagi. To zadanie jest możliwe do wykonania wobec możliwości oferowanych przez algorytmy programowania skończenieelementowego. Ponadto możliwe jest opracowanie metody numerycznej minimalizacji funkcjonału energii komplementarnej w ramach programowania wypukłego. Do takiego wniosku skłania fakt, że funkcjonał energii i funkcje ograniczające definiują zbiory wypukłe. Inną realną perspektywą jest sformułowanie i rozwiązanie zadania optymalnego projektowania ram ze sprężystoplastycznymi węzłami podatnymi. Ten wątek był poruszony w pracy [15] jedynie w zakresie sprężystej odpowiedzi węzłów. Interesującym zagadnieniem z zakresu stateczności jest opis tzw. *obszaru bezpiecznego*, to jest zbioru wartości dwu lub więcej, jednocześnie działających sił osiowych niepowodujących wyboczenia konstrukcji.

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