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Research paper

Method of modelling friction processes using piecewise-linear luz(. . .) and tar(. . .) projections

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Abstract: Friction is one of the fundamental factors describing the functioning of mechanical systems, which is why it has been the subject of extensive scientific research for years. Despite the undoubted progress in research, modeling friction processes is still an attractive scientific challenge. The article presents a method of modeling friction (kinetic and static, including the so-called “stick-slip") occurring in discrete mechanical systems. A characteristic feature of the method is the use of special luz(...) and tar(...) projections together with their original mathematical apparatus, thanks to which the friction process models are transparent, allow for parametric reduction and adapted to standard numerical procedures used in simulation. The author’s method through its application in the synthesis of an unpublished stick-slip model for a three-element system with two friction pairs. The determined model can be useful in modeling layered structures. It should be noted that analyses of stick-slip processes in layered structures can occur in the problems of soil and building mechanics.

Keywords: friction, modeling, stick-slip, three-element system

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1. Introduction

Friction is a physical phenomenon related to the dissipation of energy and counteracting the relative motion of contacting bodies or layers within one body. In tribology (as well as related materials engineering and tribotechnics), friction occurs primarily as an attribute of long-term destructive and thermodynamic processes. In theoretical and applied mechanics (soil and structural mechanics, road and rail transport, etc.), in acoustics, mechatronics and robotics, theory of nonlinear dynamics and control, friction occurs primarily as an attribute of short-term processes and stick-slip phenomena in discrete systems. The interest of researchers in the problem of friction is evidenced by numerous publications, including extensive review papers, both general ones, e.g. [1–5], and those narrowed to a specific topic, e.g. [6–11].

The classical theory of friction is based on the "macroscopic" view and simple 1D physical models of elements in friction pairs treated as rough rigid bodies with possible wetting (Coulomb–Amontons concepts and laws). Using this theory, the models of motion of discrete mechanical systems with friction initially have the form of differential inclusions, and finally – the form of ordinary differential equations with variable structures describing stick-slip processes. The structures of these models results from the balance of the inertia force, the friction force and the resultant force of other interactions causing motion. In the states of motion (when slippage of elements in friction pairs occurs), the characteristic of the kinetic friction force $F_T(V)$ is used (where V – relative velocity of moving elements in a friction pair), while in the phase of immobility (when the elements are joined) – the characteristic of the developed static friction force $F_T(F_W)$ (where F_W is the sum of the remaining external forces acting). In the simplest form, the characteristics $F_T(V)$ and $F_T(F_W)$ have piecewise-linear forms (Fig. 1).

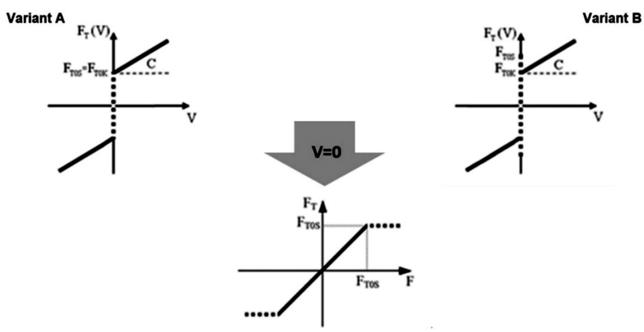

Fig. 1. Characteristics of kinetic friction forces (A and B) and static friction forces

The kinetic friction force characteristic from Fig. 1 in variant A (often called the Coulomb characteristic) refers to the basic friction model, when the limit values of the dry static and kinetic friction forces are equal ($F_{T0S} = F_{T0K} = F_{T0}$), and the viscous friction is positive and constant, defined by the damping coefficient C. The model with the Coulomb characteristic is the simplest way of expressing the essence of friction in the dynamics of mechanical systems. Therefore, it appears in publications much more often than other friction models.

The characteristic from Fig. 2 in variant B highlights the possibility of the occurrence of different ranges of the dry static friction force and the dry kinetic friction force. The so-called friction coefficient connecting the friction force with the pressure force turns out to be usually greater for static friction than for kinetic friction. The characteristic of static friction valid for zero slip velocity expresses the effect of the friction pair elements' interlocking, which ends when the friction force developed as a result of the external force reaches the allowable range. The classical theory is still the canon of theoretical mechanics and related sciences. On its basis, it is possible to model the processes accompanying the action of friction, including the stick-slip, in dynamic systems. Due to certain mathematical peculiarities, this theory is currently of particular interest to mathematicians (for example, problems of existence of solutions non-smooth systems [12], problems of bifurcations and chaos [13]). The development of MBS (MultiBody Systems) simulation software covering multi-element dynamic systems promotes the development of possibly simple piecewise-linear models [14–16].

Friction theories developed in the 20th century use also more advanced "macroscopic" 1D physical models of frictional associations, which also results in more advanced forms of mathematical models. For example: in the Stribeck model, a transition phase with negative damping appears in the characteristics of the kinetic friction force, in the Dahl model, there is an additional description of the deformation of the introduced "brush" in the friction pair, and in the latest LuGre models, even descriptions of the dynamics of elastoplastic elements introduced in the friction pair. In some publications, there are also characteristics of kinetic friction with asymmetry for negative and positive sliding velocities, and even with a different form (hysteresis) for the increase and decrease of the velocity. More advanced macroscopic friction models allow for a more accurate description of friction effects in some situations. For example, taking into account the Stribeck effect with negative damping in the kinetic friction characteristic allows for the simulation of high-frequency self-excited vibrations accompanying friction, perceived as squeaks during the braking process. Friction theories, which use advanced physical models of frictional associations, are present in various "object-oriented" models of friction systems, when, in addition to solid elements, there are elements that require a continuous description. An example of this are models of friction in the tire-road system [7, 8].

When undertaking modelling of mechanical systems, we are faced with the problem of choosing the geometric space in which the processes occurring in them will be described. Depending on the geometric configuration of the system and the details of the phenomenon under consideration, it may be necessary to model the processes in 1D, 2D or 3D space. The fact that the friction force depends directly on the normal force (Coulomb's law), and this always occurs in the description of the collision, results in the need to treat friction and collision processes together in a spatial approach (both 3D and 2D). The spatial representation of friction processes is an open problem and a difficult challenge for both the theory of mechanics and the creators of the MBS (MultiBody Systems) simulation software. Theoretical problems are primarily related to the so-called Painleve paradox [17–19]. Analyzing the motion of a rigid rod sliding with friction on a rigid substrate, Painleve found that for certain conditions of the initial angle of inclination and large values of the friction coefficient, the theoretical model based on Coulomb's law shows a mathematical inconsistency. Is it possible that Coulomb's law is not a fundamental principle for theoretical mechanics? Or maybe the assumption of absolute stiffness is a false assumption from the mathematical point of view? [11]). The introduction of susceptibility in the contact layer (implemented, among others, in the Dahl model) allows us to avoid theoretical inconsistency in friction models in 2D or 3D systems, but this in fact leads us away from the classical theory according to Coulomb's rules. Another huge challenge is the algorithmization of numerical calculations in multi-element systems. In the modeling and numerical simulation of contact processes in multi-element spatial structures, the method based on the LCP (Linear Complementarity Problem) concept is becoming increasingly popular [15, 20], in which at each computational step a certain simplified optimization problem is solved with algebraic-differential constraints resulting from the laws of mechanics (including Coulomb's friction law). However, the use of this method requires computers with high computing power.

The application of the classical friction theory in 1D, although it does not lead to any formal paradoxes, still involves a number of difficult theoretical and numerical issues. Theoretical problems concern primarily the modeling of stick-slip processes in complex systems, when, according to their structure, sticking phenomena can occur simultaneously in several friction pairs. The problem of indeterminacy of static friction forces that occurs then turns out to be bypassed by using the variational Gauss principle in the stick-slip model [10, 21]. Unfortunately, the determined stick-slip models of complex systems ultimately take on complicated formulas of hybrid equations. These models contain differential equations with a variable structure, the variability of which is controlled according to logical dependencies resulting from the states of zeroing the slip velocity in friction pairs. Difficult numerical problems are mainly concerned with "catching" during simulations of these singular zero velocity states, in which a transition from the slip phase to the stick phase can occur. Application of the Karnopp concept [22] extending the zero state to the "close to zero" state allows in many cases to avoid the situation of numerical instability. The numerical problem mentioned here occurs in the simulation of the stick-slip process, when, according to Coulomb's law, the kinetic friction characteristic contains a discontinuity for zero slip velocity.

The method of modeling friction and stick-slip processes in discrete mechanical systems proposed by the Author, using special piecewise-linear luz(. . .) and tar(. . .) projections and the Gauss principle, can be an attractive tool in the synthesis of friction models and stick-slip processes in complex discrete mechanical systems [21, 23]. It should be clearly emphasized that the developed method of modeling friction processes in discrete dynamic systems using the luz(. . .) and tar(. . .) projections does not lead to any qualitatively new models, but only expresses in a new way mathematical forms that must appear in such models in accordance with the classical theory of friction and the principles of theoretical mechanics.

The main objective of this article is to present in detail the author's method through its application in the synthesis of an unpublished stick-slip model for a three-element system with two friction pairs. The determined model can be useful in modeling layered structures.

2. The concept of using the luz(. . .) and tar(. . .) projections in modelling friction systems

According to classical theory, the characteristics of the kinetic and static friction forces $F_{T(V)}$ and $F_T(F_W)$ have piecewise-linear forms (Fig. 1). These forms can be expressed analytically in various ways, including using appropriate combinations of the linear function $y = x$, the nonlinear function $y = |x|$ and the pseudofunction $y = \text{sgn}(x)$. The concept of defining special piecewise-linear maping arose when the author modelled the dynamics of the vehicle steering system, taking into account the backlash in the gear and the friction in the wheel king-pins. Due to the topological similarities of the characteristics of the elastic reaction forces and the friction forces, common piecewise-linear projections $y = \text{luz}(x, a)$ and $y = \text{tar}(x, a)$ were formulated for them, which, as it turned out later, have an original mathematical apparatus that is very useful in analytical operations. The definitional formulas and topological interpretations of the luz(. . .) and tar(. . .) are shown in Fig. 2.

$$\text{luz}(x,a) = x + \frac{|x-a| - |x+a|}{2} \qquad \text{tar}(x,a) = x + a\cdot s(x) \quad \text{where } s(x) = \begin{cases} -1 & \text{if } x < 0 \\ s^* \in [-1,1] & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

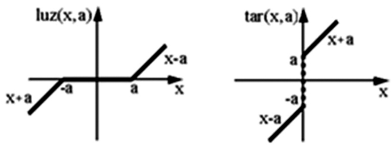

Fig. 2. luz(. . .) and tar(. . .) projections. The definition is valid for $-\infty < x < +\infty$ and a $\geqslant 0$ [23]

These projections have a number of properties expressed by formulas that create a certain mathematical apparatus useful, among others, in the synthesis, transformation and reduction of models describing friction in dynamic systems. They are derived in earlier author publications. Selected exemplary properties are presented in Fig. 3. The most important property that connects both mappings is of course the property of their mutual reversibility. Other properties concern linear combinations, compositions, formulas resolving equations, inclusions and even optimization tasks.

- $\text{luz}(x,a) = \text{tar}^{-1}(x,a)$ $\qquad$ $\text{tar}(x,a) = \text{luz}^{-1}(x,a)$
- $k\cdot\text{luz}(x,a) = \text{luz}(k\cdot x, k\cdot a)$ $\qquad$ $k\cdot\text{tar}(x,a) = \text{tar}(k\cdot x, k\cdot a)$
- $\text{luz}(\text{luz}(x,a),b) = \text{luz}(x,a+b)$ $\qquad$ $\text{tar}(\text{tar}(x,a),b) = \text{tar}(x,a+b)$
- $k_1\cdot\text{tar}(x,a_1) + k_2\cdot\text{tar}(x,a_2) = (k_1+k_2)\cdot\text{tar}\left(x, \frac{k_1\cdot a_1 + k_2\cdot a_2}{k_1+k_2}\right)$
- If $\text{luz}(y,b) = k\cdot\text{luz}(x-y,a)$ then $\text{luz}(y,b) = \frac{k}{k+1}\cdot\text{luz}(x,a+b)$
- The inclusion $0 \in y - b\cdot\text{tar}(x,a)$ is equivalent to the equation $x = \text{luz}\left(\frac{y}{b}, a\right)$
- If $f(x)$ is a convex function and $\tilde{x}: \min\limits_x f(x)$ then the task of minimization with constraints $\hat{x}: \min\limits_x f(x) \wedge |x| \leq x_0$ has the solution $\hat{x} = \tilde{x} - \text{luz}(\tilde{x}, x_0)$

Fig. 3. Selected formulas of the mathematical apparatus. All parameters here are non-negative [23]

Using the definitions of the luz(. . .) and tar(. . .) and their mathematical apparatus, one can formulate an analytical description of many linear characteristics with respect to the intervals. The examples presented in Fig. 4 refer to the most common symmetric characteristics with respect to the origin of the coordinate system (x,y). Using linear combinations of formulas with shifted arguments, one can obtain analytical forms also for asymmetric characteristics.

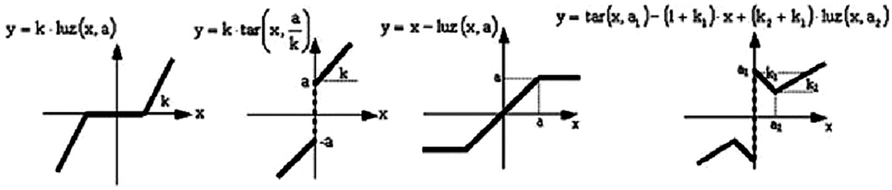

Fig. 4. Examples of analytical description of characteristics using luz(. . .) and tar(. . .) [23]

The use of the luz(. . .) / tar(. . .) in the modeling of discrete dynamic systems with friction is not only a concise expression of analytical formulas approximating the characteristics of kinetic and static friction, but also the possibility of synthesis and transformation of stick-slip models, including their reduction in a parametric way. This is illustrated in Fig. 5 for the model of a system with one moving element.

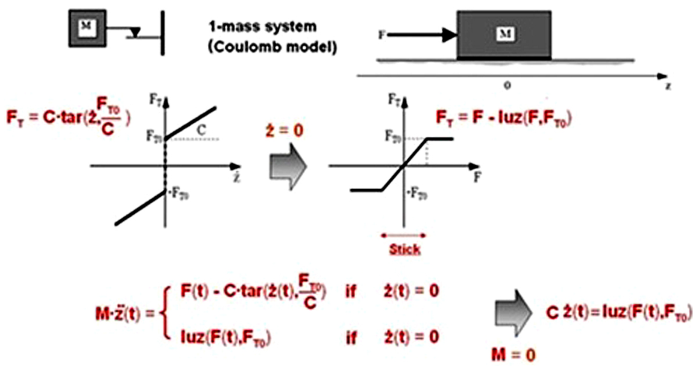

Fig. 5. The stick-slip model for a single-element system and its parametric reduction when $M = 0$ [23]

The synthesis of the friction model for a single-element system is based on analytical formulas of the characteristics of kinetic and static friction forces. In multi-element systems, when we have only assumed characteristics of kinetic friction forces in friction pairs (dependence of friction forces on sliding velocity), the matter becomes significantly more complicated. This concerns primarily the determination of static friction forces in states of zero relative velocities. The developed author's method of synthesis of friction models in multi-element systems uses the Gauss principle and the mathematical apparatus of the luz(. . .) and tar(. . .). Fig. 6 presents the structures of friction systems for which friction models have already been determined according to the developed method.

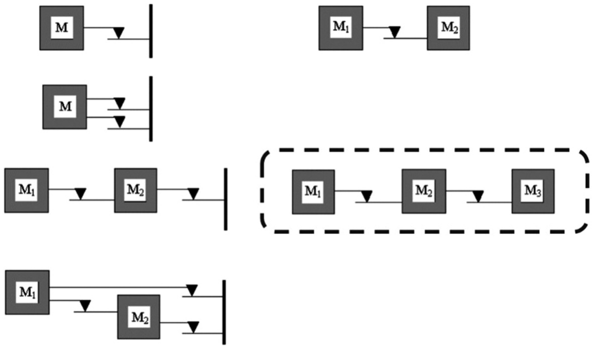

Fig. 6. Structures of friction systems for which models were determined using the author's method. The oval indicates the structure for which the synthesis of the previously unpublished model is presented in the next chapter

3. Determining the friction model on the example of a three-element system with two friction pairs

The system under consideration (Fig. 7) consists of three blocks with masses M_1, M_2 and M_3, which can move along the $0z$ axis. The blocks are affected by external forces $F_1(t)$, $F_2(t)$ and $F_3(t)$, and additionally (mutually) friction forces $F_{T12}(t)$ and $F_{T23}(t)$. The kinetic friction forces are described by Coulomb characteristics with parameters C_{12}, C_{23}, F_{T012} and F_{T023}, respectively (as in Fig. 1, in variant A). Note, in an analogous way, friction processes in a sliding bearing can be expressed (when the system is described in polar coordinates).

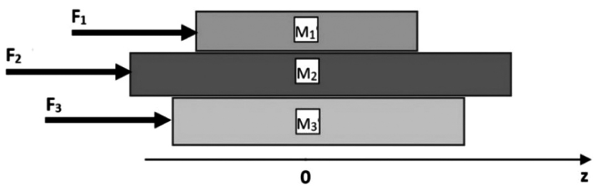

Fig. 7. Three-element system with two friction pairs

The determination of the friction model is performed in several steps described below.

Step 1 – Define the initial model in an inclusive form.

The initial mathematical model of the system consists of 3 differential inclusions:

$$M_1 \cdot \ddot{z}_1(t) \in F_1(t) - C_{12} \cdot \mathrm{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)), \frac{F_{T012}}{C_{12}}\right) \tag{3.1}$$

$$M_2 \cdot \ddot{z}_2(t) \in F_2(t) + C_{12} \cdot \mathrm{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)), \frac{F_{T012}}{C_{12}}\right) - C_{23} \cdot \mathrm{tar}\left((\dot{z}_2(t) - \dot{z}_3(t)), \frac{F_{T023}}{C_{23}}\right) \tag{3.2}$$

$$M_3 \cdot \ddot{z}_3(t) \in F_3(t) + C_{23} \cdot \text{tar}\left((\dot{z}_2(t) - \dot{z}_3(t)) \, , \frac{F_{T023}}{C_{23}} \right) \tag{3.3}$$

where:

$$s^*_{12}(t) \in [-1, 1], \quad s^*_{23}(t) \in [-1, 1] \tag{3.4}$$

The singularities $s^*_{12}(t)$ oraz $s^*_{23}(t)$ refer to states when the relative velocities in the friction characteristics are zeros.

Step 2 – Writing out the model for each variant of the relationship between speeds (4 variants):

1. When $\dot{z}_1(t) - \dot{z}_2(t) \neq 0$, and $\dot{z}_2(t) - \dot{z}_3(t) \neq 0$, the model consists of 3 equations:

$$M_1 \cdot \ddot{z}_1(t) = F_1(t) - C_{12} \cdot \text{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)) \, , \frac{F_{T012}}{C_{12}} \right) \tag{3.5}$$

$$\begin{aligned} M_2 \cdot \ddot{z}_2(t) = F_2(t) &+ C_{12} \cdot \text{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)) \, , \frac{F_{T012}}{C_{12}} \right) \\ &- C_{23} \cdot \text{tar}\left((\dot{z}_2(t) - \dot{z}_3(t)) \, , \frac{F_{T023}}{C_{23}} \right) \end{aligned} \tag{3.6}$$

$$M_3 \cdot \ddot{z}_3(t) = F_3(t) + C_{23} \cdot \text{tar}\left((\dot{z}_2(t) - \dot{z}_3(t)) \, , \frac{F_{T023}}{C_{23}} \right) \tag{3.7}$$

2. When $\dot{z}_1(t) - \dot{z}_2(t) = 0$, and $\dot{z}_2(t) - \dot{z}_3(t) \neq 0$, then – 2 inclusions and 1 equation:

$$M_1 \cdot \ddot{z}_1(t) \in F_1(t) - F_{T012} \cdot s^*_{12}(t) \tag{3.8}$$

$$M_2 \cdot \ddot{z}_2(t) \in F_2(t) + F_{T012} \cdot s^*_{12}(t) - C_{23} \cdot \text{tar}\left((\dot{z}_2(t) - \dot{z}_3(t)) \, , \frac{F_{T023}}{C_{23}} \right) \tag{3.9}$$

$$M_3 \cdot \ddot{z}_3(t) = F_3(t) + C_{23} \cdot \text{tar}\left((\dot{z}_2(t) - \dot{z}_3(t)) \, , \frac{F_{T023}}{C_{23}} \right) \tag{3.10}$$

3. When $\dot{z}_1(t) - \dot{z}_2(t) \neq 0$, and $\dot{z}_2(t) - \dot{z}_3(t) = 0$, than – 2 inclusions and 1 equation:

$$M_1 \cdot \ddot{z}_1(t) = F_1(t) - C_{12} \cdot \text{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)) \, , \frac{F_{T012}}{C_{12}} \right) \tag{3.11}$$

$$M_2 \cdot \ddot{z}_2(t) \in F_2(t) + C_{12} \cdot \text{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)) \, , \frac{F_{T012}}{C_{12}} \right) - F_{T023} \cdot s^*_{23}(t) \tag{3.12}$$

$$M_3 \cdot \ddot{z}_3(t) \in F_3(t) + F_{T023} \cdot s^*_{23}(t) \tag{3.13}$$

4. When $\dot{z}_1(t) - \dot{z}_2(t) = 0$, and $\dot{z}_2(t) - \dot{z}_3(t) = 0$, then – 3 inclusions:

$$M_1 \cdot \ddot{z}_1(t) \in F_1(t) - F_{T012} \cdot s^*_{12}(t) \tag{3.14}$$

$$M_2 \cdot \ddot{z}_2(t) \in F_2(t) + F_{T012} \cdot s^*_{12}(t) - F_{T023} \cdot s^*_{23}(t) \tag{3.15}$$

$$M_3 \cdot \ddot{z}_3(t) \in F_3(t) + F_{T023} \cdot s^*_{23}(t) \tag{3.16}$$

$s^*_{12}(t)$ and $s^*_{23}(t)$ are unknown, but can be determined individually for each velocity variant.

Step 3 – Determination of the singularity $s^*_{ij}(t)$ for each model variant based on the Gauss principle by minimizing the so-called acceleration energy Q as a function of s^*_{ij}, with constraints on its variables $s^*_{ij} \in [-1, 1]$.

In determining the singularity, the problem of minimizing the function $Q(s^*_{ij})$ without constraints is solved first by calculating and equating its appropriate derivatives to zero. On this basis $\widetilde{s}^*_{ij}$. are determined. Further using the properties of the mathematical apparatus luz(. . .) and tar(. . .), the solution of the optimization problem with constraints results from the formula:

$$\hat{s}^*_{ij} = \widetilde{s}^*_{ij} - \mathrm{luz}\left(\widetilde{s}^*_{ij}, 1\right) \tag{3.17}$$

In our case, three variants (second, third and fourth) need to be analyzed. And so:

– When $\dot{z}_1(t) - \dot{z}_2(t) = 0$ and $\dot{z}_2(t) - \dot{z}_3(t) \neq 0$, the problem concerns determining only the singularity $s^*_{12}(t)$. The minimization problem (with the constraint) has then the formula:

$$\hat{s}^*_{12} : \min_{s^*_{12}} \left(Q\left(s^*_{12}\right) = \frac{M_1\ddot{z}_1\left(s^*_{12}\right)^2 + M_2\ddot{z}_2\left(s^*_{12}\right)^2 + M_3\ddot{z}_3^2}{2} \right) \bigwedge s^*_{12}(t) \in [-1, 1] \tag{3.18}$$

Since:

$$M_1\ddot{z}_1\left(s^*_{12}\right)^2 = \frac{\left(F_1 - F_{T012}\cdot s^*_{12}\right)^2}{M_1} \tag{3.19}$$

$$M_2\ddot{z}_2\left(s^*_{12}\right)^2 = \frac{\left(F_2 - C_{23}\cdot \mathrm{tar}\left(\left(\dot{z}_2 - \dot{z}_3\right), \frac{F_{T023}}{C_{23}}\right) + F_{T012}\cdot s^*_{12}\right)^2}{M_2} \tag{3.20}$$

$$M_3\ddot{z}_3^2 = \frac{\left(F_3 + C_{23}\cdot \mathrm{tar}\left(\left(\dot{z}_2 - \dot{z}_3\right), \frac{F_{T023}}{C_{23}}\right)\right)^2}{M_3} \tag{3.21}$$

$$Q\left(s^*_{12},\right) = \frac{\left(F_1 - F_{T012}\cdot s^*_{12}\right)^2}{2M_1} + \frac{\left(F_2 - C_{23}\cdot \mathrm{tar}\left(\left(\dot{z}_2 - \dot{z}_3\right), \frac{F_{T023}}{C_{23}}\right) + F_{T012}\cdot s^*_{12}\right)^2}{2M_2} + \frac{\left(F_3 + C_{23}\cdot \mathrm{tar}\left(\left(\dot{z}_2 - \dot{z}_3\right), \frac{F_{T023}}{C_{23}}\right)\right)^2}{2M_3} \tag{3.22}$$

$$\frac{\partial Q\left(s^*_{12}\right)}{\partial s^*_{12}} = \frac{-F_{T012}\cdot\left(F_1 - F_{T012}\cdot s^*_{12}\right)}{M_1} + \frac{F_{T012}\cdot\left(F_2 - C_{23}\cdot \mathrm{tar}\left(\left(\dot{z}_2 - \dot{z}_3\right), \frac{F_{T023}}{C_{23}}\right) + F_{T012}\cdot s^*_{12}\right)}{M_2} = \frac{\left(M_1+M_2\right)F^2_{T012}}{M_1M_2}\left(s^*_{12} - \frac{M_2F_1 - M_1F_2 + M_1C_{23}\mathrm{tar}\left(\left(\dot{z}_2-\dot{z}_3\right), \frac{F_{T023}}{C_{23}}\right)}{\left(M_1+M_2\right)F_{T012}}\right) = 0 \tag{3.23}$$

Hence:

$$\widetilde{s_{12}^*} = \frac{M_2F_1 - M_1F_2 + M_1C_{23}\text{tar}\left((\dot{z}_2 - \dot{z}_3), \frac{F_{T023}}{C_{23}}\right)}{(M_1 + M_2)\, F_{T012}} \tag{3.24}$$

$$\widehat{s_{12}^*} = \frac{M_2F_1 - M_1F_2 + M_1C_{23}\text{tar}\left((\dot{z}_2 - \dot{z}_3), \frac{F_{T023}}{C_{23}}\right)}{(M_1 + M_2)\, F_{T012}} - \text{luz}\left(\frac{M_2F_1 - M_1F_2 + M_1C_{23}\text{tar}\left((\dot{z}_2 - \dot{z}_3), \frac{F_{T023}}{C_{23}}\right)}{(M_1 + M_2)\, F_{T012}}, 1\right) \tag{3.25}$$

Finally, the formula of static friction force in the state $\dot{z}_1(t) - \dot{z}_2(t) = 0$, $\dot{z}_2(t) - \dot{z}_3(t) \neq 0$:

$$F_{T012}s_{12}^*(t) = \frac{M_2F_1(t) - M_1F_2(t) + M_1C_{23}\text{tar}\left((\dot{z}_2(t) - \dot{z}_3(t)), \frac{F_{T023}}{C_{23}}\right)}{M_1 + M_2} - \text{luz}\left(\frac{M_2F_1(t) - M_1F_2(t) + M_1C_{23}\text{tar}\left((\dot{z}_2(t) - \dot{z}_3(t)), \frac{F_{T023}}{C_{23}}\right)}{M_1 + M_2}, F_{T012}\right) \tag{3.26}$$

– When $\dot{z}_1(t) - \dot{z}_2(t) \neq 0$ and $\dot{z}_2(t) - \dot{z}_3(t) = 0$, the problem concerns the determination of only the singularity $s_{23}^*(t)$.
The minimization problem (with constraint) then has the formula:

$$\hat{s}_{23}^* : \left(Q\left(s_{23}^*\right) = \frac{M_1\ddot{z}_1^2 + M_2\ddot{z}_2\left(s_{23}^*\right)^2 + M_3\ddot{z}_3\left(s_{23}^*\right)^2}{2}\right) \bigwedge s_{23}^*(t) \in [-1, 1] \tag{3.27}$$

In this case:

$$M_1\ddot{z}_1^2 = \frac{\left(F_1 - C_{12}\cdot\text{tar}\left((\dot{z}_1 - \dot{z}_2), \frac{F_{T012}}{C_{12}}\right)\right)^2}{M_1} \tag{3.28}$$

$$M_2\ddot{z}_2\left(s_{23}^*\right)^2 = \frac{\left(F_2 + C_{12}\cdot\text{tar}\left((\dot{z}_1 - \dot{z}_2), \frac{F_{T012}}{C_{12}}\right) - F_{T023}\cdot s_{23}^*\right)^2}{M_2} \tag{3.29}$$

$$M_3\ddot{z}_3\left(s_{23}^*\right)^2 = \frac{\left(F_3 + F_{T023}\cdot s_{23}^*\right)^2}{M_3} \tag{3.30}$$

$$Q\left(s_{23}^*\right) = \frac{\left(F_1 - C_{12}\cdot\text{tar}\left((\dot{z}_1 - \dot{z}_2), \frac{F_{T012}}{C_{12}}\right)\right)^2}{2M_1} + \frac{\left(F_2 + C_{12}\cdot\text{tar}\left((\dot{z}_1 - \dot{z}_2), \frac{F_{T012}}{C_{12}}\right) - F_{T023}\cdot s_{23}^*\right)^2}{2M_2} + \frac{\left(F_3 + F_{T023}\cdot s_{23}^*\right)^2}{2M_3} \tag{3.31}$$

$$\frac{\partial Q\left(s_{23}^*\right)}{\partial s_{23}^*} = \frac{-F_{T023}\cdot\left(F_2 + C_{12}\cdot\operatorname{tar}\left((\dot z_1-\dot z_2),\frac{F_{T012}}{C_{12}}\right) - F_{T023}\cdot s_{23}^*\right)}{M_2} + \frac{F_{T023}\cdot\left(F_3+F_{T023}\cdot s_{23}^*\right)}{M_3} = \frac{(M_2+M_3)F_{T023}F_{T023}}{M_1M_2}\left(s_{23}^* - \frac{M_3F_2-M_2F_3+M_3C_{12}\operatorname{tar}\left((\dot z_1-\dot z_2),\frac{F_{T012}}{C_{12}}\right)}{(M_2+M_3)F_{T023}}\right) = 0 \tag{3.32}$$

Hence:

$$\widetilde{s_{23}^*} = \frac{M_3F_2 - M_2F_3 + M_3C_{12}\operatorname{tar}\left((\dot z_1-\dot z_2),\frac{F_{T012}}{C_{12}}\right)}{(M_2+M_3)\,F_{T023}} \tag{3.33}$$

$$\widehat{s_{12}^*} = \frac{M_3F_2 - M_2F_3 + M_3C_{12}\operatorname{tar}\left((\dot z_1-\dot z_2),\frac{F_{T012}}{C_{12}}\right)}{(M_2+M_3)\,F_{T023}} - \operatorname{luz}\left(\frac{M_3F_2 - M_2F_3 + M_3C_{12}\operatorname{tar}\left((\dot z_1-\dot z_2),\frac{F_{T012}}{C_{12}}\right)}{(M_2+M_3)\,F_{T023}}, 1\right) \tag{3.34}$$

And finally, for the state $\dot z_1(t)-\dot z_2(t)\neq 0$, $\dot z_2(t)-\dot z_3(t)=0$:

$$F_{T023}s_{23}^*(t) = \frac{M_3F_2(t) - M_2F_3(t) + M_3C_{12}\operatorname{tar}\left((\dot z_1(t)-\dot z_2(t)),\frac{F_{T012}}{C_{12}}\right)}{M_2+M_3} - \operatorname{luz}\left(\frac{M_3F_2(t)-M_2F_3(t)+M_3C_{12}\operatorname{tar}\left((\dot z_1(t)-\dot z_2(t)),\frac{F_{T012}}{C_{12}}\right)}{M_2+M_3}, F_{T023}\right) \tag{3.35}$$

– When $\dot z_1(t)-\dot z_2(t)=0$, and $\dot z_2(t)-\dot z_3(t)=0$, the problem concerns the determination of $s_{12}^*(t)$ and $s_{23}^*(t)$. The minimization problem (with constraint) then has the formula:

$$\hat s_{12}^*, \hat s_{23}^* : \min_{\hat s_{12}^*,\hat s_{23}^*}\left(Q\left(s_{12}^*,\hat s_{23}^*\right) = \frac{M_1\ddot z_1\left(s_{12}^*\right)^2 + M_2\ddot z_2\left(s_{12}^*,\hat s_{23}^*\right)^2 + M_3\ddot z_3\left(s_{23}^*\right)^2}{2}\right) \bigwedge s_{12}^*(t), s_{23}^* \in [-1,1] \tag{3.36}$$

In this case:

$$M_1\ddot z_1\left(s_{12}^*\right)^2 = \frac{\left(F_1 - F_{T012}\cdot s_{12}^*\right)^2}{M_1} \tag{3.37}$$

$$M_2\ddot z_2\left(s_{12}^*, s_{23}^*\right)^2 = \frac{\left(F_2 + F_{T012}\cdot s_{12}^* - F_{T023}\cdot s_{23}^*\right)^2}{M_2} \tag{3.38}$$

$$M_3\ddot z_3\left(s_{23}^*\right)^2 = \frac{\left(F_3 + F_{T023}\cdot s_{23}^*\right)^2}{M_3} \tag{3.39}$$

$$Q\left(s_{12}^{*}, s_{23}^{*}\right) = \frac{\left(F_1 - F_{T012}\cdot s_{12}^{*}\right)^2}{2M_1} + \frac{\left(F_2 + F_{T012}\cdot s_{12}^{*} - F_{T023}\cdot s_{23}^{*}\right)^2}{2M_2} + \frac{\left(F_3 + F_{T023}\cdot s_{23}^{*}\right)^2}{2M_3} \tag{3.40}$$

$$\frac{\partial Q\left(s_{12}^{*}, s_{23}^{*}\right)}{\partial s_{12}^{*}} = \frac{-F_{T012}\cdot\left(F_1 - F_{T012}\cdot s_{12}^{*}\right)}{M_1} + \frac{F_{T012}\cdot\left(F_2 - F_{T023}\cdot s_{23}^{*} + F_{T012}\cdot s_{12}^{*}\right)}{M_2} = F_{T012}\frac{-M_2F_1+M_1F_2-M_1F_{T023}s_{23}^{*}+(M_1+M_2)F_{T012}s_{12}^{*}}{M_1M_2} = 0 \tag{3.41}$$

$$\left(M_1 + M_2\right) F_{T012}s_{12}^{*} - M_1F_{T023}s_{23}^{*} = M_2F_1 - M_1F_2 \tag{3.42}$$

$$\frac{\partial Q\left(s_{12}^{*}, s_{23}^{*}\right)}{\partial s_{23}^{*}} = \frac{-F_{T023}\cdot\left(F_2 - F_{T023}\cdot s_{23}^{*} + F_{T012}\cdot s_{12}^{*}\right)}{M_2} + \frac{F_{T023}\cdot\left(F_3+F_{T023}\cdot s_{23}^{*}\right)}{M_3} = F_{T023}\frac{-M_3F_2+M_2F_3-M_3F_{T012}s_{12}^{*}+(M_2+M_3)F_{T023}s_{23}^{*}}{M_2M_3} = 0 \tag{3.43}$$

$$-M_3F_{T012}s_{12}^{*} + \left(M_2+M_3\right) F_{T023}s_{23}^{*} = M_3F_2 - M_2F_3 \tag{3.44}$$

We solve a system of two equations. In matrix notation:

$$\begin{bmatrix} \left(M_1 + M_2\right) F_{T012} & -M_1F_{T023} \\ -M_3F_{T012} & \left(M_2+M_3\right) F_{T023} \end{bmatrix} \cdot \begin{bmatrix} s_{12}^{*} \\ s_{23}^{*} \end{bmatrix} = \begin{bmatrix} M_2F_1 - M_1F_2 \\ M_3F_2 - M_2F_3 \end{bmatrix} \tag{3.45}$$

$$\begin{bmatrix} s_{12}^{*} \\ s_{23}^{*} \end{bmatrix} = \begin{bmatrix} \left(M_1 + M_2\right) F_{T012} & -M_1F_{T023} \\ -M_3F_{T012} & \left(M_2+M_3\right) F_{T023} \end{bmatrix}^{-1} \cdot \begin{bmatrix} M_2F_1 - M_1F_2 \\ M_3F_2 - M_2F_3 \end{bmatrix} \tag{3.46}$$

Hence:

$$\widetilde{s_{12}^{*}} = \frac{\left(M_2 + M_3\right) F_1 - M_1\left(F_2 + F_3\right)}{\left(M_1 + M_2+M_3\right) F_{T012}} \tag{3.47}$$

$$\widetilde{s_{23}^{*}} = \frac{M_3\left(F_1 + F_2\right) - \left(M_1 + M_2\right) F_3}{\left(M_1 + M_2+M_3\right) F_{T123}} \tag{3.48}$$

The solutions obtained here significantly minimize $Q\left(s_{12}^{*}, \hat{s}_{23}^{*}\right)$, which was also confirmed by calculating the positive value of the Hessian:

$$\begin{vmatrix} \dfrac{\partial Q^2\left(s_{12}^{*}, \hat{s}_{23}^{*}\right)}{\partial\left(s_{12}^{*}\right)^2} & \dfrac{\partial Q^2\left(s_{12}^{*}, \hat{s}_{23}^{*}\right)}{\partial\left(s_{12}^{*}\right)\partial\left(s_{23}^{*}\right)} \\ \dfrac{\partial Q^2\left(s_{12}^{*}, \hat{s}_{23}^{*}\right)}{\partial\left(s_{23}^{*}\right)\partial\left(s_{12}^{*}\right)} & \dfrac{\partial Q^2\left(s_{12}^{*}, \hat{s}_{23}^{*}\right)}{\partial\left(s_{23}^{*}\right)^2} \end{vmatrix} = \begin{vmatrix} \dfrac{\left(M_1 + M_1\right) F_{T012}^2}{M_1M_2} & \dfrac{-F_{T012}F_{T023}}{M_2} \\ \dfrac{-F_{T012}F_{T023}}{M_2} & \dfrac{\left(M_2 + M_3\right) F_{T023}^2}{M_2M_3} \end{vmatrix} > 0 \tag{3.49}$$

Finally, for the state $\dot{z}_1(t) - \dot{z}_2(t) = 0$, $\dot{z}_2(t) - \dot{z}_3(t) = 0$:

$$F_{T012} \cdot s_{12}^{*}(t) = \frac{(M_2 + M_3)\, F_1(t) - M_1\, (F_2(t) + F_3(t))}{M_1 + M_2 + M_3} - \mathrm{luz}\left(\frac{(M_2 + M_3)\, F_1(t) - M_1\, (F_2(t) + F_3(t))}{M_1 + M_2 + M_3}, F_{T012}\right) \tag{3.50}$$

$$F_{T023} \cdot s_{23}^{*}(t) = \frac{M_3\, (F_1(t) + F_2(t)) - (M_1 + M_2)\, F_3(t)}{M_1 + M_2 + M_3} - \mathrm{luz}\left(\frac{M_3\, (F_1(t) + F_2(t)) - (M_1 + M_2)\, F_3(t)}{M_1 + M_2 + M_3}, F_{T023}\right) \tag{3.51}$$

Step 4 – Formulating the model as a system of differential equations with a variable structure

The final form of the friction model for a three-mass system with two friction pairs is expressed by a system of three differential equations with a variable structure. The analytical formulas obtained here are quite verbose, therefore in this article we refer to the numbers of the formulas used in the presentation of the model.

$$M_1 \cdot \ddot{z}_1(t) = \begin{cases} (3.5) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) \neq 0,\ \dot{z}_2(t) - \dot{z}_3(t) \neq 0 \\ (3.8), (3.25) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) = 0,\ \dot{z}_2(t) - \dot{z}_3(t) \neq 0 \\ (3.11) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) \neq 0,\ \dot{z}_2(t) - \dot{z}_3(t) = 0 \\ (3.14), (3.50), (3.51) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) = 0,\ \dot{z}_2(t) - \dot{z}_3(t) = 0 \end{cases} \tag{3.52}$$

$$M_2 \cdot \ddot{z}_2(t) = \begin{cases} (3.6) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) \neq 0,\ \dot{z}_2(t) - \dot{z}_3(t) \neq 0 \\ (3.9), (3.25) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) = 0,\ \dot{z}_2(t) - \dot{z}_3(t) \neq 0 \\ (3.12), (3.33) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) \neq 0,\ \dot{z}_2(t) - \dot{z}_3(t) = 0 \\ (3.15), (3.50), (3.51) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) = 0,\ \dot{z}_2(t) - \dot{z}_3(t) = 0 \end{cases} \tag{3.53}$$

$$M_3 \cdot \ddot{z}_3(t) = \begin{cases} (3.7) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) \neq 0,\ \dot{z}_2(t) - \dot{z}_3(t) \neq 0 \\ (3.10) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) = 0,\ \dot{z}_2(t) - \dot{z}_3(t) \neq 0 \\ (3.13), 3.33) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) \neq 0,\ \dot{z}_2(t) - \dot{z}_3(t) = 0 \\ (3.16), (3.50), (3.51) & \text{if } \dot{z}_1(t) - \dot{z}_2(t) = 0,\ \dot{z}_2(t) - \dot{z}_3(t) = 0 \end{cases} \tag{3.54}$$

Of course, the differential equations according to formulas (3.52), (3.53), (3.54) must be supplemented with initial conditions for the moment $t = 0$. In the case when at the initial moment of analysis ($t = 0$) the system is in a state of dynamic equilibrium, all initial conditions (for variables and their time derivatives) are zeros. Then, the accelerations also have zero values, which means that all friction pairs are in the stick state. In the stick state, the values of the projections luz(...) occurring in the model are zero. Precipitation from this state can occur after the appropriate action of external forces. The mechanism of entering the slip state occurs practically when the description of the dynamics of a given friction pair reaches the limit values of the static friction forces. The analysis of the determined formulas of the mathematical model allows us to capture various peculiar situations.

A key feature of models based on the luz(. . .) and tar(. . .) is their analytical nature and regularity, even though the final equations exhibit variable structures. Thanks to this regularity, it is possible to transform and reduce the model in a parametric way. Model reduction is often necessary in the simulation of so-called "numerically rigid" systems (when elements with very different masses occur in a multi-element system). This will now be explained using the example of a reduction from a three-element system to a degenerate two-element system, when in the "lower" element $M_3 \to \infty$. The physical model is shown in Fig. 8.

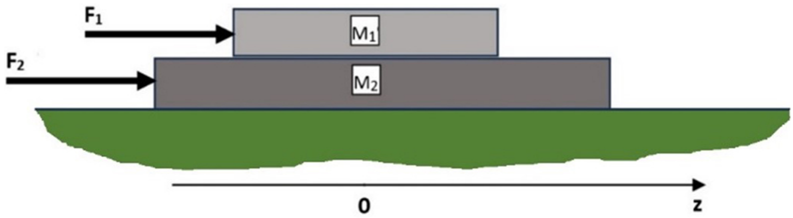

Fig. 8. Two-element system with two friction pairs

Parametric reduction is as follows:

Note that in this case $\ddot{z}_3(t) = 0$, $\dot{z}_3(t) = 0$. Now, the 4 variants of the initial model are:

1. When $\dot{z}_1(t) - \dot{z}_2(t) \neq 0$, and $\dot{z}_2(t) \neq 0$:

$$M_1 \cdot \ddot{z}_1(t) = F_1(t) - C_{12} \cdot \mathrm{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)), \frac{F_{T012}}{C_{12}}\right) \tag{3.55}$$

$$M_2 \cdot \ddot{z}_2(t) = F_2(t) + C_{12} \cdot \mathrm{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)), \frac{F_{T012}}{C_{12}}\right) - C_{23} \cdot \mathrm{tar}\left(\dot{z}_2(t), \frac{F_{T023}}{C_{23}}\right) \tag{3.56}$$

2. When $\dot{z}_1(t) - \dot{z}_2(t) = 0$, and $\dot{z}_2(t) \neq 0$:

$$M_1 \cdot \ddot{z}_1(t) \in F_1(t) - F_{T012} \cdot s^*_{12}(t) \tag{3.57}$$

$$M_2 \cdot \ddot{z}_2(t) \in F_2(t) + F_{T012} \cdot s^*_{12}(t) - C_{23} \cdot \mathrm{tar}\left(\dot{z}_2(t), \frac{F_{T023}}{C_{23}}\right) \tag{3.58}$$

3. When $\dot{z}_1(t) - \dot{z}_2(t) \neq 0$, and $\dot{z}_2(t) = 0$:

$$M_1 \cdot \ddot{z}_1(t) = F_1(t) - C_{12} \cdot \mathrm{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)), \frac{F_{T012}}{C_{12}}\right) \tag{3.59}$$

$$M_2 \cdot \ddot{z}_2(t) \in F_2(t) + C_{12} \cdot \mathrm{tar}\left((\dot{z}_1(t) - \dot{z}_2(t)), \frac{F_{T012}}{C_{12}}\right) - F_{T023} \cdot s^*_{23}(t) \tag{3.60}$$

4. When $\dot{z}_1(t) - \dot{z}_2(t) = 0$, and $\dot{z}_2(t) = 0$:

$$M_1 \cdot \ddot{z}_1(t) \in F_1(t) - F_{T012} \cdot s^*_{12}(t) \tag{3.61}$$

$$M_2 \cdot \ddot{z}_2(t) \in F_2(t) + F_{T012} \cdot s^*_{12}(t) - F_{T023} \cdot s^*_{23}(t) \tag{3.62}$$

The singularities $s^*_{12}(t)$ and $s^*_{23}(t)$ are unknown and they they are determined (for each variant individually) as for the three-element model. It can be easily shown that the obtained formulas constitute the corresponding limit transitions at $M_3 \to \infty$. So, for example, when $\dot{z}_1(t) = \dot{z}_2(t) = 0$ (variant 4):

$$\widetilde{s^*_{12}} = \lim_{M_3\to\infty} \left(\frac{(M_2 + M_3)\, F_1 - M_1\, (F_2 + F_3)}{(M_1 + M_2 + M_3)\, F_{T012}} \right) = F_1(t) \tag{3.63}$$

$$\widetilde{s^*_{23}} = \lim_{M_3\to\infty} \left(\frac{M_3\, (F_1 + F_2) - (M_1 + M_2)\, F_3}{(M_1 + M_2 + M_3)\, F_{T023}} \right) = F_1(t) + F_2(t) \tag{3.64}$$

Limiting ourselves to presenting the reduced model in the singular state $\dot{z}_1(t) = \dot{z}_2(t) = 0$ (when in dynamic equilibrium conditions we have a state of total stick state), we obtain:

$$M_1 \cdot \ddot{z}_1(t) = \text{luz}\,(F_1(t), F_{T012}) \tag{3.65}$$

$$M_2 \cdot \ddot{z}_2(t) = -\,\text{luz}\,(F_1(t), F_{T012}) + \text{luz}\,(F_1(t) + F_2(t), F_{T023}) \tag{3.66}$$

They explain the stick state and its transition to the slip state. The stick state persists until the forces $F_1(t)$ and $F_2(t)$ exceed the values F_{T012} and F_{T023}, respectively.

4. Summary

The method of modeling friction (kinetic and static, including the so-called "stick-slip") in discrete mechanical systems has been presented on the extended model. A characteristic feature of the method is the use of the piecewise-linear luz(...) and tar(...) projections together with their mathematical apparatus. Thanks to this, the friction process models are transparent, and adapted to standard numerical procedures, i.e. without entanglements. Using the luz(. . .) and tar(. . .) projections, it is possible to analytically transform models, including their parametric reductions. Of course, this method is particularly useful in relation to the classical theory of friction. However, the proposed apparatus can also be used when modeling is based on more advanced friction theories that are sometimes necessary (e.g. when Painlewe paradoxes appear).

The developed models of a three-element system with two friction pairs are ready tools for use in the simulation of layered structures. It should be noted that analyses of friction phenomena in layered structures can occur in the problems of soil and building constructions mechanics. This statement can be made by analyzing scientific publications on mechanical structures and the role of friction in civil engineering, e.g. [24, 25].

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References

[1] R.A. Ibrahim, "Friction-induced vibration, chatter, squeal, and chaos: Part I-Mechanics of contact and friction, Part II-Dynamics and modeling", *Applied Mechanics Reviews*, vol. 47, pp. 209–253, 1994.

[2] J. Awrejcewicz and P. Olejnik, "Analysis of Dynamic Systems With Various Friction Laws", *Applied Mechanics Reviews*, vol. 58, no. 6, pp. 389–411, 2005, doi: 10.1115/1.2048687.

[3] E.J. Berger, "Friction modeling for dynamic system simulation", *Applied Mechanics Reviews*, vol. 55, no. 6, pp. 535–577, 2002, doi: 10.1115/1.1501080.

[4] B. Feeny, A. Guran. N. Hinrichs, and K. Popp, "A historical review on dry friction and stick-lip phenomena", *Applied Mechanics Reviews*, vol. 51, no. 5, pp. 321–341, 1998, doi: 10.1115/1.3099008.

[5] E. Pennestrì, V. Rossi, P. Salvini, and P.P. Valentini, "Review and comparison of dry friction force models", *Nonlinear Dynamics*, vol. 83, no. 4, pp. 1785–1801, 2016, doi: 10.1007/s11071-015-2485-3.

[6] B. Armstrong-Helouvry, P. Dupont, and C. Canudas de Wit, "A survey of models, analysis tools and compensation methods for control of machines with friction", *Automatica*, vol. 30, no. 7, pp. 1083–1138, 1994, doi: 10.1016/0005-1098(94)90209-7.

[7] J. Svendenius, B. Wittenmark, *Review of Wheel Modeling and Friction Estimation*. Technical Reports, TFR 7607. Lund Institute of Technology, 2003.

[8] E. Velenis, P. Tsiotras, C. Canudas-de-Wit, and M. Sorine, "Dynamic Tire Friction Models for Combined Longitudinal and Lateral Vehicle Motion", *Vehicle System Dynamics*, vol. 43, no. 1, pp. 3–29, 2005, doi: 10.1080/00423110412331290464.

[9] C. Glocker and F. Pfeiffer, "Multiple Impacts with Frictions in Rigid Multibody Systems", *Nonlinear Dynamics*, vol. 7, pp. 471–497, 1995, doi: 10.1007/BF00121109.

[10] W. Grzesikiewicz, "Dynamics of mechanical systems with constraints", *Prace Naukowe Politechniki Warszawskiej. Mechanika,* vol. 117, 1997 (in Polish).

[11] D.E. Stewart, "Rigid-Body Dynamics with Friction and Impact", *SIAM Review*, vol. 42, no. 1, pp. 3–39, 2000, doi: 10.1137/S0036144599360110.

[12] S.J. Kim and I.J. Ha, "On the existence of Caratheodory Solutions in Mechanical Systems with Friction", *IEEE Transactions on Automatic Control*, vol. 44, no. 11, pp. 2086–2089, 1999.

[13] J. Awrejcewicz and C.H. Lamarque, *Bifurcation and Chaos in Nonsmooth Mechanical Systems*. World Scientific Publishing Company, 2003.

[14] W. Blajer and A. Markiewicz, "The effect of friction on multibody dynamics", *European Journal of Mechanics – A/Solids*, vol. 14, no. 5, pp. 807–825, 1995.

[15] B. Brogliato, T. Dam, L. Paoli, F. Génot, and M. Abadie, "Numerical simulation of finite dimensional multibody nonsmooth mechanical systems", *Applied Mechanics Reviews*, vol. 55, no. 2, pp. 107–150, 2002.

[16] M. Wojtyra, "Modeling of static friction in closed-loop kinematic chains – Uniqueness and parametric sensitivity problems", *Multibody System Dynamics*, vol. 39, no. 4, pp. 337–361, 2017, doi: 10.1007/s11044-016-9535-6.

[17] F. Génot and B. Brogliato, "New Results of Painlevé Paradoxes", *European Journal of Mechanics – A/Solids*, vol. 18, no. 4, pp. 653–678, 1999, doi: 10.1016/S0997-7538(99)00144-8.

[18] R.I. Leine, B. Brogliato, and H. Nijmeijer, "Periodic motion and bifurcations induced by the Painlevé Paradox", *European Journal of Mechanics – A/Solids*, vol. 21, no. 5, pp. 869–896, 2002, doi: 10.1016/S0997-7538(02)01231-7.

[19] A. Peiret, J. Kovecses, and J.M. Font-Llagunes, "Analysis of Friction Coupling and the Painlevé Paradox in Multibody Systems", *Multibody System Dynamics*, vol. 45, no. 3, pp. 361–378, 2019, doi: 10.1007/s11044-018-09656-y.

[20] C. Glocker and C. Studer, "Formulation and Preparation for Numerical Evaluation of Linear Complementarity Systems in Dynamics", *Multibody System Dynamics*, vol. 13, pp. 447–463, 2005, doi: 10.1007/s11044-005-2519-6.

[21] D. Żardecki, "Static friction indeterminacy problems and modeling of stick-slip phenomenon in discrete dynamic systems", *Journal of Theoretical and Applied Mechanics*, vol. 45, no. 2, pp. 289–310, 2007.

[22] D. Karnopp, "Computer Simulation of Stick-Slip Friction in Mechanical Dynamic Systems", *Transactions of the ASME. Journal of Dynamic Systems, Measurement, and Control*, vol. 107, no. 1, pp. 100–103, 1985.

[23] D. Żardecki, "Piecewise linear luz(. . .) and tar(. . .) projections. Part 1 – Theoretical background, Part 2 – Application in modeling of dynamic systems with freeplay and friction", *Journal of Theoretical and Applied Mechanics*, vol. 44, no. 1, pp. 163–202, 2006.

[24] M. Świeczek, "In-plane friction of elastic bodies undergoing kinematic quasi-static excitations", *Archives of Civil Engineering*, vol. 53, no. 1, pp. 23–55, 2007.

[25] M. Gołdyn, "Shear capacity of the interface between concretes cast at different time in the light of experimental investigations and codes of practice", *Archives of Civil Engineering*, vol. 68, no. 1, pp. 275–298, 2022, doi: 10.24425/ace.2022.140168.

Metoda modelowania procesów tarcia z wykorzystaniem odwzorowań przedziałami-liniowych luz(. . .) i tar(. . .)

Słowa kluczowe: tarcie, modelowanie, stick-slip, układ trójelementowy

Streszczenie:

Tarcie jest jednym z podstawowych czynników opisujących funkcjonowanie układów mechanicznych, dlatego od lat jest przedmiotem szeroko zakrojonych badań naukowych. Mimo niewątpliwego postępu w badaniach, modelowanie procesów tarcia jest nadal atrakcyjnym wyzwaniem naukowym. W artykule przedstawiono metodę modelowania tarcia (kinetycznego i statycznego, w tym tzw. „stick-slip") występującego w dyskretnych układach mechanicznych. Cechą charakterystyczną metody jest wykorzystanie specjalnych odwzorowań luz(...) i tar(...) wraz z ich oryginalnym aparatem matematycznym, dzięki czemu modele procesów tarcia są przejrzyste, pozwalają na redukcję parametryczną i są dostosowane do standardowych procedur numerycznych. Autorska metoda jest zaprezentowana w syntezie niepublikowanego modelu stick-slip dla układu trójelementowego z dwiema parami ciernymi. Opracowany model może być przydatny w modelowaniu struktur warstwowych. Należy zauważyć, że analizy procesów stick-slip w strukturach warstwowych mogą występować w problemach mechaniki gruntów i budowli.

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