WARSAW UNIVERSITY OF TECHNOLOGY Index 351733
FACULTY OF CIVIL ENGINEERING
COMMITTEE FOR CIVIL AND WATER ENGINEERING
POLISH ACADEMY OF SCIENCES ISSN 1230-2945

DOI: 10.24425/ace.2026.158617

ARCHIVES OF CIVIL ENGINEERING

Vol. LXXII ISSUE 2 2026

pp. 405 – 418

Research paper

Scheduling of construction projects based on Z^*-numbers

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Abstract: The article addresses the challenge of scheduling construction projects given the considerable uncertainty associated with project time estimation. A fuzzy approach, which allows for modelling uncertainty, is employed to represent the duration of project tasks. It is shown, however, that the most well-known fuzzy numbers do not encompass all aspects of uncertainty. This study thus introduces a novel application of a specific type of fuzzy numbers, the Z^*-numbers, which allow for the consideration of several factors that strongly influence the estimation process, including individual human characteristics, experience, and the project and task context. The underlying theory is discussed, and its application to the scheduling of construction projects is proposed. The approach and its benefits are demonstrated through a practical example involving the construction of a single-family house. Advantages include a more credible and accurate estimation of task durations that accounts for uncertainty while mitigating subjective, individual, and context-dependent elements. Further research perspectives are also outlined.

Keywords: construction project, context, fuzzy set, risk, uncertainty, Z-number

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1. Introduction

Construction projects are characterised by very high complexity and are inherently subject to numerous risks. These risks often manifest in substantial and frequent delays of construction projects [1–3]. It is estimated that such delays concern about 60% of construction projects [4]. For this reason, it is important to investigate methods of modelling task duration in construction projects. The goal is to achieve, during the project planning phase, the most accurate reflection of the actual situation and knowledge about risk and its potential impact on the ultimate project duration. This would ensure that the schedules used for project management are as accurate and credible as possible, allowing for a comprehensive analysis of project delay risks.

In the literature, various approaches to the modelling of the risk linked to the duration of tasks of projects, including construction projects, are utilised. The most common approach is probabilistic modelling, e.g. [5]. The probabilistic approach, however, is not always suitable for projects, as it largely relies on historical data that are not always available. The fuzzy approach is an important and significant alternative, often used by researchers to model project risk, including in construction projects, e.g. [6–9]. The quality of fuzzy number modelling, however, strongly depends on the experience, knowledge, and other characteristics of the human modeler (alternatively called estimator or evaluator), as well as the general context. For this reason, the novel concepts of Z-fuzzy numbers and further, Z^*-fuzzy numbers have been introduced in the literature. Z-numbers largely avoid the problem of modeler dependence, as they take into account the modeler credibility, assessed on the basis of their past performance. Z^*-fuzzy numbers additionally account for the context of the project and its tasks, as well as other (e.g., emotional) factors influencing the estimation process, which is ignored in the probabilistic and other fuzzy risk modelling approaches. While Z-numbers have been used in project risk management, their application to construction projects is rare, and Z^*-fuzzy numbers have never been applied in the project management framework.

Therefore, the aim of this paper is to propose, for the first time in the literature, an approach to project risk modelling based on Z^*-fuzzy numbers. This approach will be dedicated to construction projects, though it is not limited to them and can be applied to other project types too. A real-world single-family construction project schedule is used to initially verify the proposed approach. The outline of the paper is as follows: In Section 2, we present basic theoretical information concerning fuzzy numbers, with special emphasis on Z- and Z^*-fuzzy numbers. In Section 3, basic conclusions on the state of art are presented regarding the usage of fuzzy numbers in project scheduling, with an emphasis on construction projects, as well as Z- and Z^*-fuzzy numbers. Section 4 describes the proposal of applying Z^*-fuzzy numbers to the scheduling of construction projects, using a known application of Z-numbers in the same context as a starting point. Section 5 initially verifies the proposal using a real-world construction project. The paper concludes with some final remarks.

2. Fuzzy numbers, and their special types: Z- and Z^*-numbers

Fuzzy numbers, a special case of fuzzy sets (fuzzy numbers are defined on the set of real numbers, while fuzzy sets may be defined on arbitrary sets), were introduced in the 1960s by Lotfi A. Zadeh and have found countless applications in various fields. There are classic fuzzy numbers as well as various other types [10–12]. Triangular fuzzy numbers [13], are a good representative of classic fuzzy numbers.

Definition 1:

A triangular fuzzy number $\widetilde{A} = (A_a, A_b, Ac)$ is a triple of crisp (real) numbers A_a, A_b, A_c such that $A_a < A_b < A_c$, with a continuous membership function μ_A, determined on the set of real numbers, with values belonging to the interval $[0, 1]$, such that:

- $\mu_A(x) = 0$ for $x \leq A_a$ and $x \geq A_c$,
- $\mu_A(x) = 1$ for $x = A_b$,
- $\mu_A(x)$ is linear and increasing on the interval (A_a, A_b) and linear and decreasing in the interval (A_b, A_c).

A positive triangular fuzzy number is a triangular fuzzy number with $A_a \geq 0$. A crisp fuzzy number is a special case of triangular fuzzy number. The interval $[A_a, A_c]$ is called the support of the triangular fuzzy number $\widetilde{A}$.

The fuzzy number $\widetilde{A} = (A_a, A_b, A_c)$ can be seen as the estimation of a magnitude A that is not fully known in the estimation moment. It is uniquely known that it will be "about" A_b. $\mu_A(x)$ represents the degree to which the human estimator considers it possible that A will take on value x. Values x below A_a and over A_c are considered impossible. The possibility degree is the highest, i.e., 1, for A_b, and it diminishes to 0 as the distance of x from A_b increases.

Other classic fuzzy numbers differ from triangular fuzzy numbers in the shape of the membership function. Basic fuzzy numbers are a useful tool for representing incomplete knowledge about a value, but this representation depends strongly on the estimator or evaluator, including their experience, expertise, subjective views, and personal attitudes, such as optimism or pessimism, as well as the context of the estimation.

Z-type fuzzy numbers, introduced by Zadeh in 2011, extend classic fuzzy numbers with an element that helps to reduce individual-related bias [14]. This additional element is the certainty degree of estimation [15], also known as credibility [13] or reliability [15].

Definition 2:

A Z-type fuzzy number $(\widetilde{A}, \widetilde{Z})$ (or simply a Z-number) is a fuzzy number $\widetilde{A}$ completed with a credibility assessment $\widetilde{Z}$, being also a fuzzy number.

A person distinct from the estimator judges the estimator's credibility or reliability of $\widetilde{A}$, in the form of a fuzzy number $\widetilde{Z}$, and then $\widetilde{A}$ is adjusted into $A\widetilde{Z}$ that is also a fuzzy number (Definition 1), representing the available information about possible values of the magnitude A in a more objective way. The transformation of $(\widetilde{A}, \widetilde{Z})$ into $A\widetilde{Z}$ can be done in many ways (see, e.g., [16]).

In [14], the authors go a step further and propose breaking down the element $\widetilde{Z}$ in Definition 2 into several elements: time (T), context (C), and feelings and emotions (F) of the evaluator. Such fuzzy numbers are called Z^*-fuzzy numbers. Here is their simplified definition:

Definition 3:

A Z^*-type fuzzy number (or simply a Z^*-number) $(\widetilde{A}, \widetilde{Z}, T, C, F)$ is a fuzzy number $\widetilde{A}$ completed with the following information:

- credibility assessment $\widetilde{Z}$, referring purely to the individual credibility of the modeler
- information about the time setting T of the estimation (e.g., before or after a certain milestone), can be given in a linguistic form or as a crisp or fuzzy set
- information about the context C of the estimation in moment T, can be given in a linguistic form or as a crisp or fuzzy set
- information about the feelings and emotions F of the modeler in moment T, in context C, can be given in a linguistic form or as a crisp or fuzzy set.

The Z^*-number $(\widetilde{A}, \widetilde{Z}, T, C, F)$ is then transformed into a more objective version of $\widetilde{A}$, denoted as $A\widetilde{Z}$, using a predefined method based on the elements $\widetilde{Z}, T, C, F$.

The aspects of time and emotions are important in scheduling, e.g., in IT projects, where customers often change their requirements, but they seem to be less critical in construction projects, in which the scope is usually better defined [17,18]. Therefore, we will use Definition 3 without including the elements of time (T) and feelings (F).

The concept of context (C) is very broad and can be understood in various ways. However, many authors mention context as a crucial element influencing uncertainty linked to the process of scheduling project tasks because context largely determines project risks [19,20].

In [20], the context of software development projects is considered, which consists of the following elements: application domain, used technology, application type, project duration, team experience. In [21], factors influencing project context are summarized. Three internal factors were distinguished: (a) organizational factors, e.g., number of project stakeholders, number of projects implemented simultaneously); (b) work content factors, e.g., length of tasks, amount of resources needed, number of changed requirements, quality problems; (c) factors related to resource availability. Additionally, 5 external factors were classified: (a) logistics and purchasing factors; (b) environmental factors (e.g., weather); (c) sociopolitical factors (e.g., changes in health regulations or government tax policies); (d) market factors (e.g. exchange rate changes, material costs); technological factors (e.g. technological complexity or technological uncertainty, such as the introduction of new materials or methods). The context of construction projects will be described in Section 4.

Before we present our proposal, in the following section we summarize the results of the literature search on the usage of fuzzy numbers in construction projects management, especially scheduling.

3. Fuzzy sets in construction project management – state of art

In a systematic literature review on the usage of fuzzy sets in construction project management, it was found out that practically only classic fuzzy sets are applied. Other types of fuzzy sets (intuitionistic, hesitant, type-2, etc.) have been used in a very limited number of cases. Z-numbers were are only in [13,22]. The two aforementioned papers apply Z-numbers to project scheduling. That concept will be discussed further, as it forms the basis of our proposal. According to the literature review, Z^*-numbers have never been used in project management.

4. Application of fuzzy numbers to project scheduling

In this section, we will take the first step to address the gap identified in the previous section by proposing an application of Z^*-numbers to the scheduling of construction projects. This proposal builds on the only existing application of Z-numbers to the scheduling of projects general. Consequently, we will first outline the existing approach, and then describe the new one.

4.1. Summary of the existing application of Z-numbers to project scheduling

Let $\widetilde{A} = \left(A_a^j, A_b^j, A_c^j\right)$ represent the planned duration of a project task in the estimation phase, with the actual duration being unknown and uncertain. Suppose this estimation was given by a person frequently asked to serve as an expert in estimating the duration of specific project tasks. The project manager, in order to assess the credibility of the estimate $\widetilde{A}$, examines the historical records of this particular expert. Let us suppose there are N records, composed of the following elements:

– $\widetilde{A}_j = \left(A_a^j, A_b^j, A_c^j\right)$ – estimated value in the j-th record, $j = 1, \ldots, N$,

– $\widetilde{R}_j = \left(R_a^j, R_b^j, R_c^j\right)$ – real (actual) value in the j-th record, $j = 1, \ldots, N$.

Obviously, the actual values $\widetilde{R}_j = \left(R_a^j, R_b^j, R_c^j\right)$ will usually be crisp values, but we allow for the possibility of incomplete knowledge about the actual task duration even after the task is finished (for example, in the case of research activities, which are difficult to measure due to the imprecise nature of creative processes). Additionally, if the actual duration of the same task type has varied substantially in the past, we can take R_b as the average task execution time in the task history, R_a (R_c) as the shortest (longest) execution time, and set $\left(R_a^j, R_b^j, R_c^j\right) = (R_a, R_b, R_c)$ for all $j = 1, \ldots, N$.

In practice also estimations $\widetilde{A}_j$ are often given as crisp numbers [5], in which case we assume $A_a = A_b = A_c$ and $A_a^j = A_b^j = A_c^j$ for all $j = 1, \ldots, N$.

Then, the project manager determines the expert reliability $\widetilde{Z} = (Z_a, Z_b, Z_c)$ using formula (4.1), in order to generate the Z-number $\left(\widetilde{A}, \widetilde{Z}\right)$ [13].

$$\widetilde{Z} = (Z_a, Z_b, Z_c) = \left(1 + \frac{\sum_{j=1}^{N} \left(R_\alpha^j - A_a^j\right) / A_a^j}{N}, 1 + \frac{\sum_{j=1}^{N} \left(R_b^j - A_b^j\right) / A_b^j}{N}, 1 + \frac{\sum_{j=1}^{N} \left(R_c^j - A_c^j\right) / A_c^j}{N}\right) \tag{4.1}$$

Generally, the credibility would be 1 if all the planned values were equal to the actual one. In other cases, the credibility differs from 1, with values higher than 1 indicating underestimation (the actual value being greater than the planned one) and values lower than 1 indicating overestimation. Finally, an adjusted estimate $A\widetilde{Z}$ (a fuzzy number) of the task duration is generated. Details are given in [13].

4.2. Proposal of an application of Z^*-numbers to construction project scheduling

The Z^*-numbers allow to take into account the context of the estimation process. Let us now discuss the context for construction projects. Based on [20, 21] and on the basis of information about the project industry (construction industry), the following contextual dimensions can be proposed, which allow the assignment of fuzzy schedule tasks to a specific project and organizational context, affecting the uncertainty of these tasks: c_1: project budget, scale: 1 (very small) – 5 (very large), this is the characteristic of the construction project being implemented; c_2: project duration, scale: 1 (very short) – 5 (very long), this is a characteristic of the construction project being implemented; c_3: team experience, scale: 1 (very small) – 5 (very large), this is the characteristics of the project team; c_4: number of projects implemented simultaneously, scale: 1 (very few) – 5 (very many), this is the characteristic of the organization implementing the project; c_5: amount of resources needed, scale: 1 (very little) – 5 (very much), this is the characteristic of the project being implemented; c_6: influence of weather factors, scale: 1 (very small) – 5 (very large), this is a characteristic of the task, depending on the season in which a given construction task was planned, but also on the type of task performed; c_7: influence of market factors, scale: 1 (very small) – 5 (very large), this is the characteristics of the external environment, but also the characteristics of the task being performed (e.g. some building materials needed to implement certain project tasks may show large price fluctuations in a given construction season); c_8: technological complexity, scale: 1 (very small) – 5 (very large), this is the characteristic of the task being performed; c_9: degree of novelty, scale: 1 (very small) – 5 (very large), this is a characteristic of the task being performed and the choice of technology.

Having data about the context (C) and the measurability of context elements in the form of a uniform scale (1–5), the process of assessing the reliability of the estimation described in [13], and summarized in formula (4.1), can be modified to assess the reliability by means of formula (4.1), taking into account only those historical records that refer to the estimation of the tasks that had a similar context. It is possible to propose a simple method of calculating the similarity of contexts, based on the k-nearest neighbors method. Given the context of the current task represented in the form of a vector $[c_1, \ldots, c_9]$, where $c_i \in \{1, \ldots, 5\}$, one can find the k closest context vectors (where k is a fixed integer, e.g. 3) by counting the Euclidean distance of the vectors [23] and perform reliability assessment according to (4.1), taking into account the historical records only for tasks linked to the k closest contexts determined in this way.

The context-dependent approach has the advantage of disregarding past estimation inaccuracies linked to irrelevant context-related phenomena or features. For instance, it would be questionable to treat estimations performed under completely different weather conditions and weather stability in the same way. Additionally, this approach supports an efficient learning process to improve the quality of project time estimation.

Below, sample calculations leading to a fuzzy schedule are presented using the example of a "constructing the foundations" task. The input is a crisp estimation A (a special case of the triangular fuzzy number) – in this organization, the common practice of using crisp estimations has been accepted. The output is a fuzzy estimation $A\widetilde{Z}$ obtained after applying the algorithm of context-based adjustment, which is based on the history of estimations and corresponding contextual information. For simplicity, the context dimensions presented in Table 1 are reduced to three dimensions: project budget, technological complexity, and degree of novelty. It is further assumed that available historical records can be classified into four different contexts: C_1, C_2, C_3, and C_4, and that the current estimation is performed in context C_3. Using the method outlined above, the k-nearest contexts for $k = 3$ were determined, which turned out to be C_1, C_2, and, obviously, C_3. No records linked to the context C_4 will be taken into account.

Table 1. Example of the application of the proposed approach to a task „constructing the foundations"

<table>
<tr><td colspan="5">Input: Crisp estimation of the duration of "constructing the foundations" A=3 [days]</td></tr>
<tr><td colspan="5">Step 1: Calculate distance between current context and available historical contexts</td></tr>
<tr><td></td><td>Context C_1</td><td>Context C_2</td><td>Context C_3</td><td>Context C_4</td></tr>
<tr><td>Context description</td><td>Inexperienced team</td><td>More experienced team</td><td>Experienced team</td><td>New technology was introduced</td></tr>
<tr><td>Project budget (1-–5)</td><td>1</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td>Technological complexity (1–5)</td><td>1</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td>Degree of novelty (1–5)</td><td>4</td><td>3</td><td>2</td><td>5</td></tr>
<tr><td>Distance to the current context C_3</td><td>2</td><td>1</td><td>0</td><td>4.12</td></tr>
<tr><td colspan="5">Step 2: k-nearest contexts are determined: C_1, C_2, C_3 ($k = 3$). The records linked to context C_4 are ignored</td></tr>
<tr><td colspan="5">Step 3: The history of "constructing the foundations" is analyzed in k-nearest contexts</td></tr>
<tr><td></td><td>$j = 1(C_1)$</td><td>$j = 2(C_2)$</td><td>$j = 3(C_2)$</td><td>$j = 4(C_3)$</td></tr>
<tr><td>Realization $R(j)$, $j = 1, \ldots, N$</td><td>4.5 [days]</td><td>4</td><td>3</td><td>5</td></tr>
<tr><td>Estimation $A(j)$, $j = 1, \ldots, N$</td><td>3.5 [days]</td><td>3 [days]</td><td>3 [days]</td><td>4 [days]</td></tr>
<tr><td colspan="5">Step 4: Convert crisp realizations $R(j)$ to fuzzy realizations $\left(R_a^j, R_b^j, R_c^j\right) = (R_a, R_b, R_c)$ for all $j = 1, \ldots, N$</td></tr>
<tr><td></td><td>$j = 1(C_1)$</td><td>$j = 2(C_2)$</td><td>$j = 3(C_2)$</td><td>$j = 4(C_3)$</td></tr>
<tr><td>$R_a^j = \min R(j)$, $j = 1, \ldots, N$</td><td>3</td><td>3</td><td>3</td><td>3</td></tr>
<tr><td>$R_b^j = \text{Avg}\,(R(j))$, $j = 1, \ldots, N$</td><td>4.125</td><td>4.125</td><td>4.125</td><td>4.125</td></tr>
<tr><td>$R_c^j = \min R(j)$, $j = 1, \ldots, N$</td><td>5</td><td>5</td><td>5</td><td>5</td></tr>
<tr><td colspan="5">Step 5: Convert crisp estimations Ato fuzzy estimation $\widetilde{A} = (A_a, A_b, A_c)$</td></tr>
<tr><td></td><td>$j = 1(C_1)$</td><td>$j = 2(C_2)$</td><td>$j = 3(C_2)$</td><td>$j = 4(C_3)$</td></tr>
<tr><td>$A_a^j = A(j)$, $j = 1, \ldots, N$</td><td>3.5</td><td>3</td><td>3</td><td>4</td></tr>
<tr><td>$A_b^j = A(j)$, $j = 1, \ldots, N$</td><td>3.5</td><td>3</td><td>3</td><td>4</td></tr>
<tr><td>$A_c^j = A(j)$, $j = 1, \ldots, N$</td><td>3.5</td><td>3</td><td>3</td><td>4</td></tr>
<tr><td colspan="5">Step 6: Determine fuzzy credibility adjustment according to [13], formula (4.1),
but restricted to a subset of records that belong to k-nearest contexts:
$Z_a = 0.902$
$Z_b = 1.240$
$Z_c = 1.503$</td></tr>
<tr><td colspan="5">Output: $A\widetilde{Z} = (2.71, 3.72, 4.51)$ is the adjusted fuzzy estimation of „constructing the foundations"</td></tr>
</table>

In the above example, the initial estimation of task duration, 3 days, has been adjusted to a fuzzy number (2.71, 3.72, 4.51), taking into account the history of estimating the duration of the same project task type by the same expert, and the context of the estimation.

$A\widetilde{Z} = (2.71, 3.72, 4.51)$ clearly indicates that there is uncertainty linked to the duration of this task, which should be considered while analyzing and managing the project. The most probable duration, 3.72 days, is substantially higher than the initial estimate, and there is a possibility that the actual duration could be even longer.

5. Case study

Below the remaining work (for a selected status date, the 22nd of February) of an ongoing family house construction project execution is presented (Table 2).

Table 2. Remaining works to be completed

Id	Activity
22	External finishing works (external windowsills, canopies, gutters, balcony railings)-
23	Painting works, tiles
24	Kitchen and bathroom equipment
25	Internal finishing works
26	Acceptance of works

Three different scenarios will be considered when determining project end time: (1) Crisp estimations, (2) Adjustments based on Z-fuzzy numbers (with no context information), as in [13] (3) Adjustments based on Z^*-fuzzy numbers (with context information), according to the proposal from the previous section.

For crisp estimations the planned finish is estimated for 11th of April and the critical path contains activities 23, 25 and 26 (Table 3).

Table 3. House construction project remaining schedule with crisp estimations

Id	Crisp estimation *A* [days]	Critical path	Planned start	Planned finish
22	17	No	2024-02-23	2024-03-18
23	17	Yes	2024-02-23	2024-03-18
24	6	No	2024-03-19	2024-03-26
25	12	Yes	2024-03-19	2024-04-03
26	6	Yes	2024-04-04	2024-04-11

For adjustments based on Z-fuzzy numbers, the context information was ignored, and all historical estimations were used to determine the adjustments. The procedure from [13] was executed for all activities, but due to the size of this paper, only the activity with $Id = 22$ ("External finishing works") will be presented below (Table 4). The input is a crisp estimation A (equal to 17), along with historical information about the estimations and actuals of the duration of "External finishing works" tasks in projects from the past. The conversion of historical estimations and actual values to fuzzy numbers is obtained as described above.

Table 4. Historical records with fuzzy conversions for the "External finishing works" activity, without taking into account the context

j – index of the record	Estimation [days]	Actuals [days]	Fuzzy conversion of the estimation			Fuzzy conversion of the actuals		
1	18	35	18	18	18	12	25.5	40
2	17	37	17	17	17	12	25.5	40
3	16	34	16	16	16	12	25.5	40
4	18	40	18	18	18	12	25.5	40
5	18	15	18	18	18	12	25.5	40
6	16	12	16	16	16	12	25.5	40
7	16	15	16	16	16	12	25.5	40
8	17	16	17	17	17	12	25.5	40

Formula (4.1) gave the following reliability assessment: $\widetilde{Z} = (0.71, 1.50, 2.32)$, and the adjusted duration (versus the initial estimate $A = 17$): $A\widetilde{Z} = (12, 26, 40)$.

Based on these adjustments, the remaining schedule was determined, and a new critical path was calculated using fuzzy estimations obtained by multiplying the original estimations by fuzzy credibility adjustments. The critical path was then calculated by converting fuzzy numbers back to crisp values using the values having the greatest membership degree ($\widetilde{Z}_b$). The output schedule is presented in Table 5. The estimations were rounded to whole days.

Table 5. House construction project remaining schedule with Z-numbers based adjustments

Id	Initial estimation *A* [days]	Adjusted estimation	Critical path (based on $A\widetilde{Z}_b$)	Planned start	Planned finish
22	17	(12, 26, 40)	No	2024-02-23	2024-03-29
23	17	(15, 19, 19)	Yes	2024-02-23	2024-03-20
24	6	(5, 7, 8)	No	2024-03-21	2024-03-29
25	12	(11, 14, 15)	Yes	2024-03-21	2024-04-09
26	6	(4, 6, 7)	Yes	2024-04-10	2024-04-17

It should be noted that the critical path has not changed in this case, but the planned finish date has been shifted from 2024-04-11 (using crisp estimations) to 2024-04-17 (using Z-fuzzy estimations), becoming more accurate and credible.

For the context-aware fuzzy credibility adjustments (Z^*-fuzzy numbers) the historical records were filtered, based on the context information.

Let us come back to activity $Id = 22$ ("External finishing works"). In this case the context information differed between contexts at c6: influence of weather factors. In Table 4, only first four records were executed with similar weather conditions, i.e. with "weather influence of 5 (very high)". The activity of "External finishing works" is particularly sensitive for weather conditions if it is being executed during winter (as in the case of the current schedule, which starts on February 22). The historical records with the relevant context were recalculated and are presented in Table 6.

Table 6. Historical records with fuzzy conversions for the "External finishing works" activity, with context information for the "External finishing works" activity

j – index of the record	Estimation [days]	Actuals [days]	Fuzzy conversion of the estimation			Fuzzy conversion of the actuals		
1	18.00	35.00	18.00	18.00	18.00	34.00	36.50	40.00
2	17.00	37.00	17.00	17.00	17.00	34.00	36.50	40.00
3	16.00	34.00	16.00	16.00	16.00	34.00	36.50	40.00
4	18.00	40.00	18.00	18.00	18.00	34.00	36.50	40.00

The same steps as above led to $\widetilde{Z} = (Z_a, Z_b, Z_c) = (1.98, 2.12, 2.32)$, versus $\widetilde{Z} = (0.71, 1.50, 2.32)$ in the case of ignoring the context information. The credibility assessment gave substantially different results for the approach based on Z-numbers, and that based on Z^*-numbers. When the context information was ignored, the support of the fuzzy reliability assessment $\widetilde{Z}$ included 1, and even values smaller than 1, which meant that overestimation is possible for the considered task. The context analysis shows that only substantial underestimations are possible (the smallest number in the support of $\widetilde{Z}$ is almost 2). This means that the risk of delay of the considered task is much higher that we would have thought without the application of the Z^*-based approach.

As shown in Table 7, taking into account the context changes the critical path (activities 22 and 26 are now critical) as well as shifts the planned finish date for 2024-04-22, making it more credible and realistic.

Table 7. House construction project remaining schedule with Z^*-number based adjustments

Id	Initial estimation *A* [days]	Adjusted estimation	Critical path (based on $A\widetilde{Z}_b$)	Planned start	Planned finish
22	17	(34, 36, 40)	Yes	2024-02-23	2024-04-12
23	17	(15, 19, 19)	No	2024-02-23	2024-03-20
24	6	(5, 7, 8)	No	2024-03-21	2024-03-29
25	12	(11, 14, 15)	No	2024-03-21	2024-04-09
26	6	(4, 6, 7)	Yes	2024-04-15	2024-04-22

The described three different use cases show that crisp estimations can lead to an unrealistic schedule with an overly optimistic planned finish date. Fuzzy estimations incorporate information about uncertainty, which shifts the planned finish date and may change the critical path. However, they may still not accurately reflect the situation due to their dependence on human factors and context. Z-numbers, and even more so Z^*-numbers, help to eliminate or at least reduce this dependency.

In the case study presented here, it was observed that using Z-fuzzy numbers did indeed shift the end date, making it more accurate and realistic, but the critical path remained unchanged. This occurred because reliability was calculated using all available estimations, also those made in completely different contexts. This paper introduces a novel approach based on the idea of Z^*-fuzzy numbers, which uses context information for calculating credibility. We suggested a method for calculating the adjustments based on the k-nearest neighbors algorithm for context similarity, which led to more credible and accurate results.

Context can be very important for estimating uncertainties in construction projects (e.g., activities sensitive to weather conditions can be significantly delayed in some weather conditions, as happened in the described example, and remain unshifted in other weather conditions). Z^*-fuzzy calculations produced a new schedule where context-based uncertainties caused changes in the critical path. It should also be noted that the greatest differences in calculating uncertainties were observed for only one activity, "External finishing works", while other activities were not as affected. However, this one activity was crucial because it caused the change in the critical path. The resulting schedule produced the latest planned finish date compared to other methods (crisp estimation or Z-numbers credibility-adjusted estimations).

6. Conclusions

"There are no facts about the future" – these words from 1977 are becoming increasingly true in today's global and rapidly changing project environment [24]. Accurate estimation of project duration is not an easy task, including in the construction industry, because there are numerous unknowns during the estimation process [5]. For this reason it is advisable [5, 24] not to avoid risk modelling in the estimation process [25] of construction project durations, but to attempt to model it instead, making the time estimations of construction project tasks as faithful to reality as possible. One possible approach here is to use fuzzy numbers [26].

In this paper, we applied, for the first time in the literature, a special type of fuzzy numbers, the so-called Z^*-numbers, to the estimation of construction project scheduling. These fuzzy numbers allow us to model both the lack of knowledge, the personal traits of the estimator, their expertise, experience, and the context in which the estimation process is taking place. We demonstrated the potential usefulness of this approach in terms of the quality of schedule information obtained, using the case study of a small construction project, against the background of existing approaches to construction project scheduling.

There are numerous limitations of the study that indicate further research perspectives:

- We considered a reduced definition of the Z^*-number. Aspects such as feelings and emotions, or the time setting of the estimation, were not taken into account.
- We considered only one possible description of the context of construction projects and treated it as something fixed. The investigation into the context of construction projects and its unstable nature (the context may change during the project lifecycle) has not been performed.
- The concept of distance or similarity between contexts, or more broadly, between the Z^*-numbers, still needs to be researched.

– We used only a small construction project as the case study. More complex case studies are needed to verify and validate the approach.

We hope that the proposed approach will contribute to more successful and satisfactory implementation of construction projects.

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Harmonogramowanie projektów budowlanych przy użyciu liczb Z^*

Słowa kluczowe: projekt budowlany, kontekst, zbiór rozmyty, ryzyko, niepewność, liczba Z.

Streszczenie:

Artykuł omawia problematykę harmonogramowania projektów budowlanych w warunkach dużej niepewności dotyczącej czasu realizacji zadań, która to niepewność jest cechą charakterystyczną projektów budowlanych. Autorzy prezentują definicje podstawowych pojęć związanych z liczbami rozmytymi, w tym liczby trójkątnej rozmytej oraz liczb typu Z i ich rozszerzenia, liczb typu Z^*. W artykule przedstawiono również wyniki przeglądu literatury dotyczącego stosowania liczb rozmytych w zarządzaniu projektami budowlanymi. Pokazano, że liczby Z^* dotąd nie były wykorzystywane w harmonogramowaniu projektów budowlanych, a liczby typu Z w ograniczonym zakresie. Liczby typu Z i Z^* eliminują subiektywizm wynikający z pozycji lub cech osobistych osoby estymującej, ponieważ dodają ocenę wiarygodności osoby estymującej, a liczby Z^* dodatkowo uwzględniają kontekst zadania oraz ewentualne czynniki emocjonalne. W proponowanym podejściu w estymacjach, modelowanych za pomocą liczb typu Z^*, proponuje się korzystać z informacji na temat estymacji (i ich dokładności) podobnych zadań dokonywanych przez te same osoby w przeszłości, ale uwzględniając tylko zadania estymowane w podobnych kontekstach. Kontekstem może być np. pogoda, a podobne konteksty to np. podobnie niesprzyjające warunki atmosferyczne w przeszłości. Proponowane podejście jest rozszerzeniem znanego, zbliżonego podejścia stosującego liczby typu Z, ale nieuwzględniającego kontekstu. Jako

przykład ilustrujący zaproponowane podejście autorzy analizują harmonogram rzeczywistego projektu budowy domu jednorodzinnego. Za pomocą tego przykładu przedstawiono zalety zastosowania liczb Z^* do modelowania niepewności w projektach budowlanych. Autorzy wykonali symulacje dla trzech różnych scenariuszy estymacji czasu zakończenia projektu: przy wykorzystaniu estymacji dokładnych, liczb Z oraz liczb Z^* uwzględniających kontekst. Wyniki wskazują, że estymacje dokładne mogą prowadzić do nadmiernie optymistycznych harmonogramów, natomiast liczby Z i Z^* lepiej odzwierciedlają ryzyko opóźnień. Wprowadzenie kontekstu pozwalało na bardziej realistyczną ocenę ryzyka i zmianę ścieżki krytycznej projektu, przesuwając planowany termin zakończenia prac. W zakończeniu artykułu autorzy podkreślają, że dalsze badania mogą obejmować rozwój bardziej złożonych studiów przypadków, uwzględnienie dodatkowych zmiennych oraz doskonalenie metody oceny podobieństwa kontekstów.

Received: 2024-11-08, Revised: 2025-01-08