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Research paper

Inelastic column buckling mechanism

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Abstract: This paper presents attempt to explain the phenomenon of the stability loss of prismatic columns, made of metal, loaded beyond the yield limit. The explanation is presented on the base of Euler's theory, tangent modulus theory, reduced modulus theory, Shanley's column paradox and inelastic column theory. Explanation refers to results of experiments performed by many scientists, including F.R. Shanley, and related to Huber–Mises–Hencky (HMH) hypothesis of material effort. The use of HMH material effort hypothesis to analyse buckling column within inelastic range enables to explain the mechanism of buckling (including strain development) observed by F.R. Shanley and points out that the column paradox does not exist. The change of the uniform state of strain and stress to the non-uniform state which does not lead toloss of total stability by column is referred to as the loss of internal stability. The concept is presented on the base of idealised stress-strain diagram (bilinear elasto-plastic model).

Keywords: buckling mechanism, Huber–Mises–Hencky, inelastic column theory, internal stability, material effort hypothesis

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1. Introduction

The problem of bars structure stability related to bending or torsional buckling is still important and is addressed by many authors, e.g. Siedlecka [1], Błażejewski et al. [2], Giżejowski et al. [3] are just few to point out. The base for such advanced methods are the fundamental cases.

The critical parameters values which causes elastic buckling of prismatic columns made from metal is described by Euler's formula:

$$P_k = \frac{\pi^2 EI}{l_r^2} \tag{1.1}$$

in which: E – modulus of elasticity (value determined from stress-strain diagram in tensile test), I – moment of inertia of a cross section of column, l_r – buckling length of column, P_k – critical load.

The Eq. (1.1) satisfies the equilibrium of forces and moments, which act on the bent element of column. Stress distribution in column subjected to an axial load P, before the buckling begins, is uniform and has a magnitude σ_k within the cross section. Since the column starts to bend from a straight position, both the strain and the stress vary linearly and the change of stress at both sides is the same, because the relation between stress and strain is described by elastic modulus E. Stress distribution is shown in the Fig. 1.

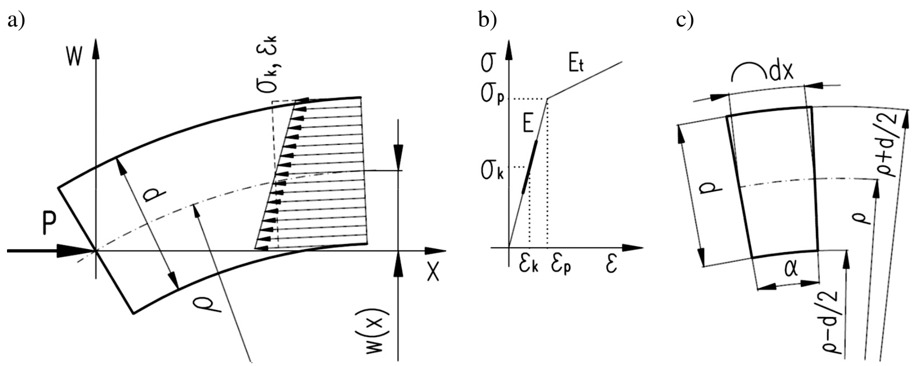

Fig. 1. Elastic buckling column mechanism (own elaboration): (a) distribution of stress in bent column, (b) illustrative stress distribution on the idealised stress-strain diagram, (c) deformed length element of column

2. Tangent modulus theory

For compressive load exerting normal stress beyond the yield limit σ_p (Fig. 1b), Euler formula (1.1) is not valid, because relation between strain and stress does not depend on E (elastic modulus) F. Engesser had proposed convenient formula (based on the Euler equation) to estimate the critical values of inelastic buckling. He had suggested to use the tangent modulus E_t instead of elastic modulus E:

$$P_k = \frac{\pi^2 E_t I}{l_r^2} \tag{2.1}$$

where: E_t – tangent modulus (value determined from stress-strain diagram in tensile test).

Eq. (2.1) does not satisfy equilibrium conditions, i.e. net force condition and net moment condition. This is empirically derived formula. This equation suggests that for stress values above the σ_p, during loss of stability, entire cross section is loaded and strain and stress is developed according to stress-strain diagram in tensile test.

Distribution of normal stress does not satisfy the basic equilibrium condition – the net force equal zero. Increase of stress across the cross section should be followed by increase of value of force P (Fig. 2).

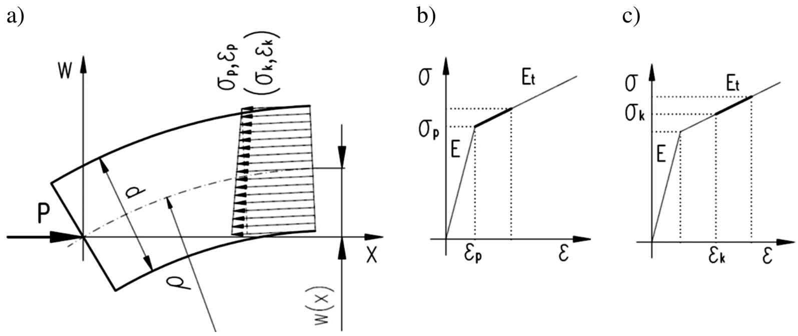

Fig. 2. Assumed inelastic buckling mechanism according to tangent modulus theory E_t (own elaboration): (a) stress distribution in bent column; origin loaded with P, which causes σ_p (or σ_k respectively) stress across the cross section, (b) and (c) distributions of normal stress according to tangent modulus theory E_t

This does not account for stress redistribution, that is, unloading on convex side of column, which other investigators had pointed out, e.g., Polish scientist F. Jasiński [4].

3. Reduced modulus theory

Th. von Karman had performed tests and developed the theoretical formula for value of reduced modulus E_r and published the results of the tests in his doctoral dissertation. This theory (known as E_r theory) satisfies both equilibrium conditions, that is, net force and net moment condition and takes loading on concave side of column and unloading on convex side of column into account. Loading area and unloading area are separated from each other by neutral line (it has a radius of curvature ρ). The normal stress at neutral line equals to σ_p (or σ_k respectively). Stress distribution is shown in the Fig. 3.

Considering equilibrium conditions of the sum of forces acting upon the bent element of column with rectangular cross section (dimensions $b \times d$), the formula which relates radius of curvature ρ of the neutral line, elastic modulus E, tangent modulus E_t and s_p coordinate, takes the form:

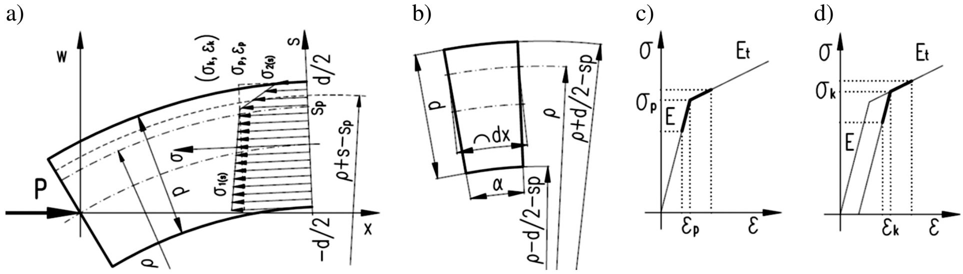

Fig. 3. Assumed inelastic buckling mechanism according to reduced modulus theory E_r (own elaboration): (a) stress distribution in bent column; originally loaded with force P, which causes stress σ_p (or σ_k respectively) across the cross section, (b) deformed length element of column, (c) and (d) distributions of normal stresses according to reduced modulus theory E_r

$$\frac{1}{\rho} \cdot \left[E\left(\frac{d}{2} - s_p\right)^2 - E_t\left(\frac{d}{2} + s_p\right)^2 \right] = 0 \tag{3.1}$$

This equation is satisfied, regardless of radius of curvature ρ (also for straight column), if the second factor (the term in square brackets) equals zero. Redistribution of stress (change of the values from uniform to non-uniform) can occur at the same value of force P, if only coordinate s_p satisfies Eq. (3.2). When redistribution of stress starts, a neutral line is located on coordinate s_p, which heavily depends on elastic modulus E, tangent modulus E_t and thickness d of column.

$$s_p = \frac{d}{2} \cdot \frac{\sqrt{E} - \sqrt{E_t}}{\sqrt{E} + \sqrt{E_t}} \tag{3.2}$$

By using the second condition of equilibrium – the net moment condition – it is possible to derive formula for reduced modulus E_r value (introduced by Th. von Karman [5]):

$$E_r = \frac{4 \cdot E \cdot E_t}{E + E_t + 2\sqrt{EE_t}} \tag{3.3}$$

and general formula for critical load:

$$P_k = \frac{\pi^2 E_r I}{l_r^2} \tag{3.4}$$

Numerous tests performed with inelastic columns had revealed that results are closer to the tangent modulus theory E_t than to reduced modulus theory E_r [6–8]. An important observation is that the tangent modulus values are lower than the experimental results [6–8], (Fig. 4). Therefore, use of tangent modulus theory for calculation of critical load of columns provides a safety margin.

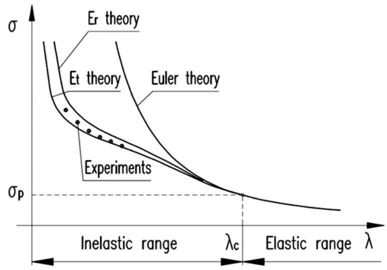

Fig. 4. Comparison of E_t and E_r theories versus test data (own elaboration); λ – slenderness ratio of column, σ – normal stress

4. Inelastic column paradox and reported strain test data according to F.R. Shanley

Since the reduced modulus theory had not been sufficiently confirmed by experiments, in 1946, Lockheed Aircraft Corporation performed further tests. The main objective was to identify the strain at unloaded side of column when load increases. The obtained experimental results were a subject of two papers published by F.R. Shanley. In the first paper [7], he formulated theses of inelastic column paradox, related to tangent modulus theory and reduced modulus theory: (1) Ad tangent modulus theory E_t. If stress at the convex side equals to value which was reached before starting bending and over entire cross section stress increases above this value (Fig. 2), the average stress has to be greater than the one generated by compressive force at constant value. (2) Ad reduced modulus theory E_r. According to this theory, column should remain unbuckled until the load reaches critical value. For the column, in order to preserve its stiffness for load above tangent modulus, the reversal strains are required. However, it is impossible to have reversal strain in straight column. Paradox theses are illustrated in Fig. 5. The expected change of strain during increase of compressive load for both the tangent modulus theory and reduced modulus theory, is depicted.

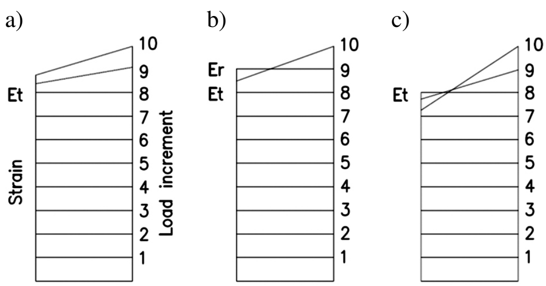

Fig. 5. Assumed strain mechanisms of columns (own elaboration on the base [8]): (a) according to tangent modulus theory E_t, (b) according to reduced modulus theory E_r, (c) expected development of strain, if load is above tangent modulus, in order to lose the stability according to reduced modulus theory E_r

In his second paper [8], in its first part, F.R. Shanley presented results of precisely measured strain at both sides of column (concave and convex) during increase of compressive load. The value of strain differ at both sides of column just at about 40000 lbs; this represents 43% of maximum load. The difference increased as value of load increased. Finally at maximum load (92500 lbs), within unloading zone, at convex side of column, some residual strain of about 0.00067 are observed. The strain distribution from experiment made by F.R. Shanley is presented in the Fig. 6.

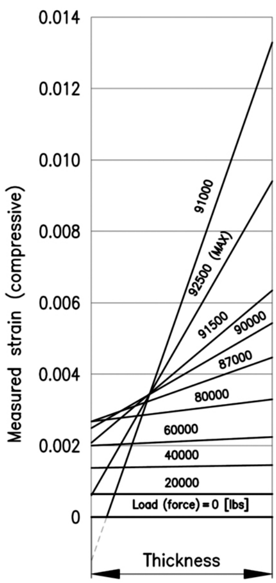

Fig. 6. Strain distribution from experiment made by F.R. Shanley (own elaboration on the base [8])

5. Shanley's theory

In the second part of the paper [8], F.R. Shanley presented his own idea on mechanism of loss of stability. He modelled the buckled column as two infinite rigid bars coupled together with joint (Fig. 7).

Otherwise than the models of column in Euler theory (elastic buckling), tangent modulus theory E_t and reduced modulus theory E_r, the model of column proposed by F.R. Shanley significantly differs from real column. The F.R. Shanley's concept of column undergoing the buckling in inelastic range was used for numerous buckling analyses of columns and plates. However, oversimplification and mathematical analysis used, make the model usage for description of mechanism of strain development very limited. The mathematical analysis is not related to dimensions of cross section of column. The analysis contains some inaccuracy, e.g., the unit of moment of force is expressed in unit of force instead of unit of moment.

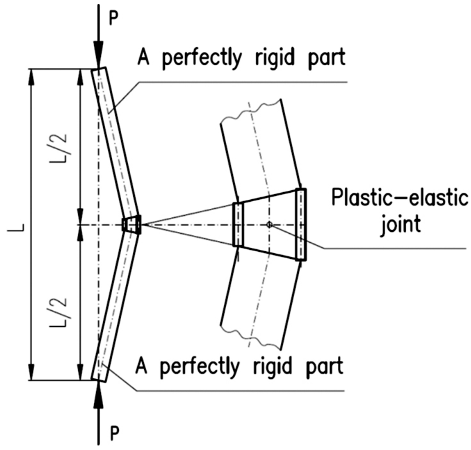

Fig. 7. Shanley's model of column (own elaboration on the base of [8])

6. About foregoing theories and assumption to column buckling mechanism

None of the foregoing inelastic buckling theory presents a theoretical mechanism of development of the strain and stress in column during buckling, and they do not explain: (1) greater accuracy of tangent modulus theory E_t (empirical formula) compared to the test data than reduced modulus theory E_r (consistent with equilibrium conditions), (2) F.R. Shanley inelastic column paradox, (3) process of strain distribution recorded by F.R. Shanley.

Although buckling beyond the yield limit was one of the first question of the mechanics of materials, which was related to plasticity of materials and was a subject of numerous test, the analysis did not take account for material effort. It was assumed that in compressed column, within inelastic range, relation between strain and stress follows the stress-strain diagram of tensile test. During the compression of column beyond the yield limit, a loading area with stress in inelastic range occurs. Within an inelastic range a small increment of stress produces big increment of strain, because plastic strains are greater than elastic (compared to the same increment of stress). Because normal stress within loaded area is greater than yield limit, thus analysis of inelastic buckling mechanism should consider effort of material.

In order to develop a more logically consistent theory, the analysis should base on equilibrium conditions and fundamentals of mechanics of materials. From theoretical point of view it is worth to notice that only reduced modulus theory E_r satisfies both the net force equilibrium and the net moment equilibrium condition. Due to plastic component of strain, within loading area, the greater increment (absolute value) of strain than decrement of strain within unloading area can be observed. Plastic strain is permanent and within loading area additional permanent slope of the cross section versus neutral line occurs.

On base of Eq. (3.1) and Eq. (3.2) one can conclude that, if the column is stocky enough, stress redistribution can occur in "straight column", i.e., in column which did not buckled as whole. During stress redistribution potential energy of force P decreases, therefore it is applied through the centroid of the cross section. The centroid of the cross section is within loaded area. On the other hand, the energy of deformation within the column volume increases.

7. Loss of internal stability

The mechanism of strain and stress redistribution in column without the loss of stability as an entire column is a kind the loss of internal stability. Figure 8 depicts the distribution of stress at the moment of loss of internal stability, as well the shape (Fig. 8b) that may have column which is stocky enough.

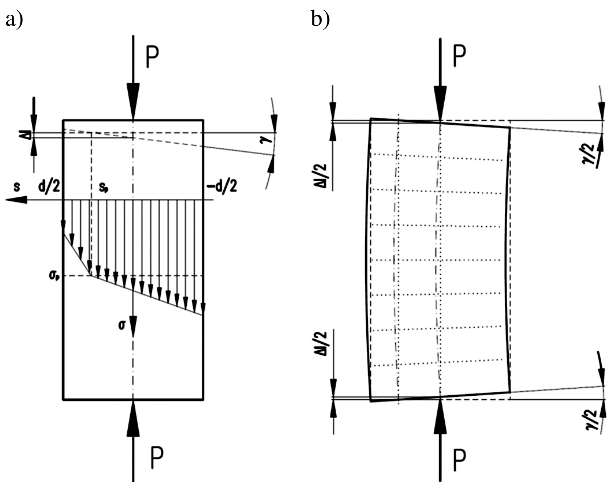

Fig. 8. Mechanism of internal loss of stability (own elaboration): (a) illustrative stress distribution in the instant of loss of internal stability, (b) possible shape of column after the process fulfilled

By Δl shortening of fibres located on axis of symmetry is denoted (the influence of bowling is neglected). Along this axis compressive force acts. Due to the slope of cross section (angle γ) a perpendicular component of force P occurs (decomposition of P on two directions). Tangential component of force creates shear stress. The effect of shear stress in elastic range of buckling on value of critical force was addressed by S.P. Thimoshenko and J.M. Gere [9]. The shear stress is not related to bending but to component of force P, thus the value of stress is the same across the sections. For the plane state of stress described by normal stress and shear stress, material effort is the best described by the Huber–Mises–Hencky (HMH) hypothesis for such material, as aluminium, steel, cooper. It was validated by tests performed by G.I. Taylor and H. Quinney in 1931 [10]. In the Fig. 9 the curve of yield boundary is depicted.

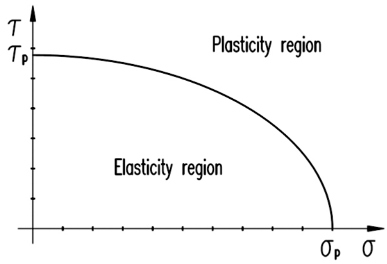

Fig. 9. Yielding criterion according to hypothesis HMH (own elaboration on the basis of [10])

The state of stress on the inner region enclosed by the curve is elastic, while on the other side it is plastic. The correlation between values of normal stress and values of shear stress which the yield curve follows is given by formula:

$$\left(\frac{\sigma}{\sigma_p}\right)^2 + 3 \cdot \left(\frac{\tau}{\sigma_p}\right)^2 = 1 \tag{7.1}$$

Figure 10 depicts probable effect of shear stress value (depends on angle γ) on value of normal stress inside the elastic region referred to stress-strain diagram tensile test.

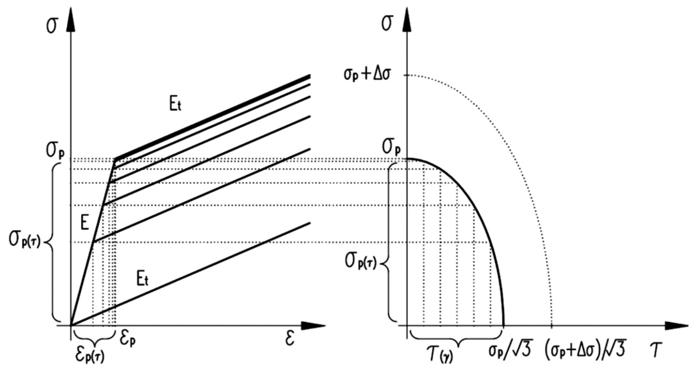

Fig. 10. Set of characteristics on idealised stress-strain diagram – influence of shear stress on normal stress at which yielding limit in space (σ, τ) is obtained (own elaboration)

The shear stress development within area where normal stress equal to or greater than plastic limit σ_p acts, changes elastic state of stress to plastic state (Fig. 9). Along with increase of shear stress the plasticisation of material follows at lower value of normal stress which belongs to elastic region.

From reduced modulus theory E_r comes that at a moment of initiation of stress redistribution neutral line is located on coordinate s_p; see Eq. (3.2) and Fig. 3a, and Fig. 8b. As the stress redistribution proceeds, generated shear stress across the cross section cause transition of the state of stress on neutral line to plastic region. Therefore, in loaded area exists the elastic normal stress which magnitude is smaller than stress magnitude at coordinate s_p. The location of neutral line is shifted in direction of unloaded zone, where elastic state of stress exists. In the Fig. 11 state of stress in column is shown at beginning of loss of internal stability and at the end.

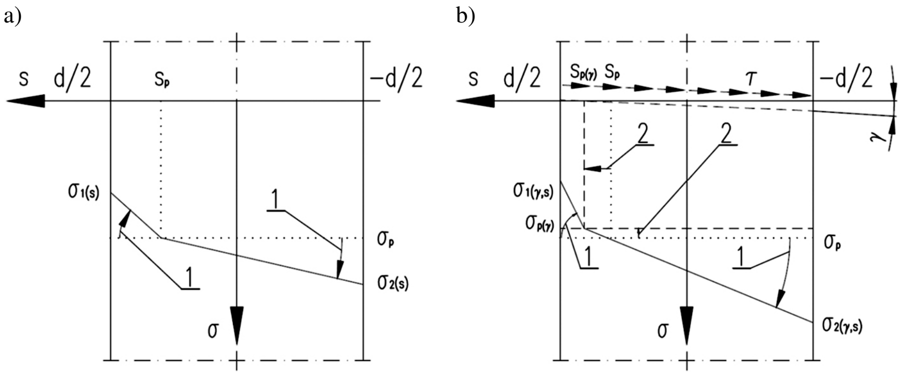

Fig. 11. Probable processing of stress redistribution during loss of internal stability (own elaboration): (a) "1" – direction of stress redistribution at the beginning, (b) "1" – direction stress redistribution shift at the end, "2" – direction of neutral line shift and decreasing the normal stress on neutral line at new location

Due to enlarging of the region with smaller value of modulus (tangent modulus E_t) the decreasing of stiffness is observed.

Figure 12 illustrates possible change of the state of normal stress during the loss of internal stability in column loaded with force P, which originally causes uniform normal stress σ_p across the cross sections. In the Fig. 12(a) change of stress distribution on the base of idealised stress-strain diagram is shown, and the 12(b) shows the same stress distribution on the base

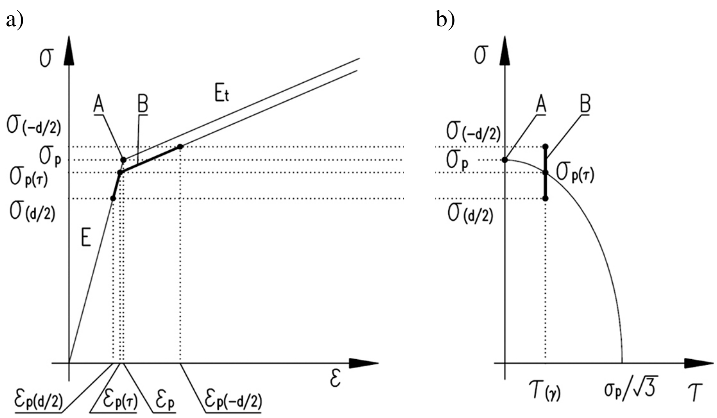

Fig. 12. Possible change in stress distribution (own elaboration): (a) presented on stress-strain diagram (b) presented on the yielding criterion diagram. Point A – state before, and line B – after loss of internal stability

of material effort diagram according to HMH hypothesis in σ, τ space. Point A represents uniform normal stress distribution in the entire column. Line B represents non-uniform normal stress distribution after the loss of internal stability process has finished. During the loss of internal stability it comes to qualitative change in stress distribution. The uniform state of normal stress converts to non-uniform state of normal stress plus shear stress.

8. Converting potential energy to deformation energy in a column-force system

Figure 13 depicts schematically the change of deformation energy in column during loss of internal stability process when column has been loaded to normal stress equal σ_P.

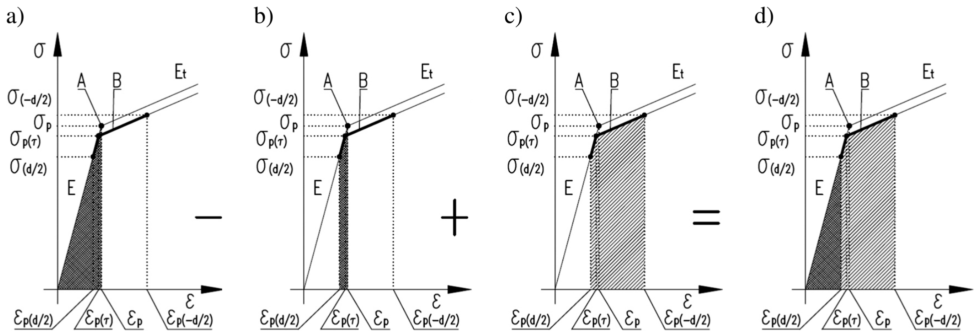

Fig. 13. The change of energy during process of loss of internal stability (own elaboration): (a) an initial state, (b) decreasing of energy in unloading zone, (c) increasing the energy related to non-uniform sate of stress, (d) state after the process has ended

The increase of strain at the axis of acting force P, decreases potential energy and increases distortion energy in column. The process of loss of internal stability probably breaks at a moment when the energy becomes in equilibrium, i.e. increase of deformation energy equals to decreasing of potential energy. During the redistribution the energy related to state of uniform normal stress decreases (Fig. 13b) and, in turn, energy related to distribution of non-uniform normal stress increases (Fig. 13c).

9. Presumable development of stress while increasing the compressive force

Taking into account the shear stress and material effort according to HMH hypothesis, it is possible to explain why for load beyond the yield limit it is not possible to reach uniform normal stress (see Fig. 6). Note that according to this concept of buckling mechanism,

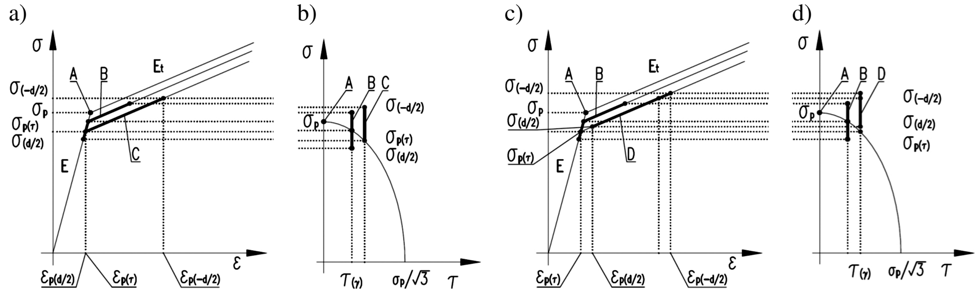

Fig. 14. Possible stress distributions in column during increase of the compressive load (own elaboration): (a) and (c) presented on stress-strain diagram, (b) and (d) presented on the yielding criterion diagram

within plastic zone it is not possible to cause the stress which reflects stress-strain diagram of tensile test. In the Fig. 14 presumable states of stress while increasing compressive force is shown.

On the assumption that column is stocky enough and is loaded, initially the uniform stress across the cross section obtains the value σ_p – point A in the Fig. 14a. Due to change of the slope of curve in the diagram the loss of internal stability occurs, followed by development of shear stress and redistribution of normal stress (depicted by line B). Figure 14b, line B is partially located in elastic region and partially in plastic region. Further increase of load will cause increase of strain, whereby significantly greater it will be at concave side than at convex one. The angle γ will increase, and as a result the value of shear stress will increase and normal stress in elastic zone decrease, until the moment of cancelling the unloading zone. The unloading zone will disappear when the value of normal stress equals to value of stress taken from material effort curve. In this case stress distribution will follow according to line E_t in the graph that is lower than for origin at stress-strain diagram; entire line C in the Fig. 14a is situated on the lower line E_t. In the Fig. 14b entire line C is situated in plastic zone. Only one its point is situated on material effort curve. If stress-strain diagram in plastic zone is linear (constant value of E_t), further increase of loading will occur with the same increment of strain at both sides of column. The angle γ will not increase. The non-uniform normal stress will increase according to line E_t. The normal stress distribution is shown as line D in Fig. 14c and Fig. 14d. The transition from stress distribution following the line C to stress distribution following the line D causes only increase of strain and stress according to line E_t. The shear stress will increase due to greater value of force, but not due to increase of the value of angle γ. The buckling column mechanism which considers material effort according to HMH hypothesis, explains why during increasing of the value of compressive force progressively larger difference between values of strain at both sides is observed (Fig. 6). After reaching the yield limit it is not possible for the uniform normal stress to exist while increasing compressive force. The development of stress can account for the tangent modulus E_t, however, it will follow the line located below the nominal curve in stress-strain diagram. From the Fig. 6 it can be read out that maximum load reported in the test was 92 500 lbs. At this maximum value of load at convex side of column (unloaded) small strain existed. It is possible that it is a plastic, irreversible strain.

10. Total loss of stability

Total loss of stability by column loaded beyond the yield limit (inelastic), depends on value of compressive force, slenderness ratio of column, parameters of stress-strain diagram in tensile test, and history of loss of internal stability. The development of strain at the moment of general loss of stability, at permanent value of force, is likely related with generating both loaded and unloaded zones. The paper presents, in a illustrative way, a case corresponding to the test described by F.R. Shanley [8] (see Fig. 6). Due to limited space, the case of simultaneous loss of internal and total stability is unmentioned. In the Fig. 15 change in stress distribution in column while increasing force is shown. The stress is changing from uniform σ_p (point A at initial value of force) to non-uniform depicted with line B. The change from B to C and, in turn to D, is a result of increasing compressive force. The state illustrated by line F exists at maximum value of compressive force. Total loss of stability is referred to as the state when column is not be able to take the greater load.

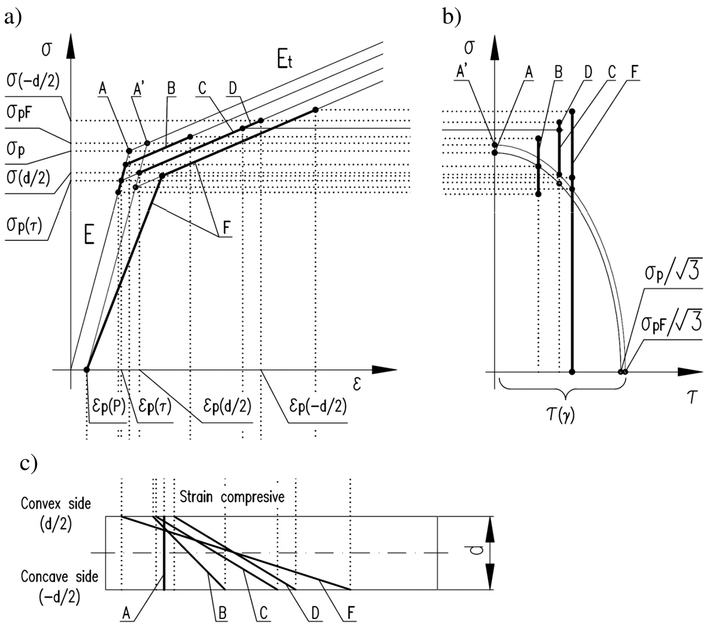

Fig. 15. Presumable stress and strain distribution during the increasing load (own elaboration): (a) distribution for set of curves on the stress-strain diagram, (b) distribution of set of curves on material effort diagram according to HMH hypothesis, (c) strain distribution

Notions in the Fig. 15 are as follows: A – point representing uniform stress equals to yield limit, B – line representing state of non-uniform stress (after loss of internal stability), C and D – lines represent stress redistribution as a result of increasing force (on line D irreversible stress does not exist, all stress are beyond the initial boundary yield limit related to σ_p – new yield limit is σ_{pF} and point A' is a value of boundary condition in HMH diagram), F – line which shows presumable stress distribution at maximum (critical) value of compressive force; in this state the column loses its stability. $\varepsilon_{p(P)}$ – value of irreversible plastic strain at convex (unloaded) side at normal stress equal zero (cf. Fig. 6), $\varepsilon_{p(t)}$ – value of irreversible plastic strain at convex side depending on value of shear stress (thereby an angle γ), $\varepsilon_{p(d/2)}$ – value of irreversible plastic strain at convex side, $\varepsilon_{p(-d/2)}$ – value of plastic strain at concave side, $\sigma_{p(t)}$ – value of normal stress at convex side depending on value of shear stress (thereby an angle γ), $\sigma_{p(d/2)}$ – value of normal stress a convex side, $\sigma_{p(-d/2)}$ – value of normal stress at concave side.

11. Summary

Foregoing column theories for inelastic range (tangent modulus and reduced modulus) take account only for normal stress in inelastic range according to stress-strain diagram in tensile test. Due to stress value beyond the yield limit material effort should be considered, e.g. according to HMH hypothesis for metal. After reaching uniform stress across the cross section equal to proportional limit (yielding stress) it is likely that in column automatically occurs stress redistribution and state of uniform normal stress changes to non-uniform. The loss of internal stability results in decreasing of bending stiffness of column. After initiating the loss of internal stability process the value of normal stress, related to elastic zone, on neutral line decreases. The examination of inelastic column stability mechanism base upon relation between stress and strain estimated in tensile test is oversimplification. This paper only presents attempt of new approach to explain the inelastic column phenomenon. In order to obtain more detailed information further research is necessary.

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Mechanizm wyboczenia prętów w zakresie sprężysto-plastycznym

Słowa kluczowe: Huber–Mises–Hencky, mechanizm wyboczenia, wewnętrzna stabilność, wpływ naprężeń stycznych, wyboczenie w zakresie plastycznym

Streszczenie:

Zagadnienie wyboczenia prętów w zakresie sprężystym dobrze opisuje teoria L. Eulera, która spełnia warunki równowagi w postaci sumy siłi sumy momentów. Mechanizm wyboczenia prętów w zakresie sprężysto-plastycznym nie jest dotychczas dostatecznie opisany. Istniejące teorie, np. teoria modułu stycznego E_t, nie spełnia warunków równowagi (sumy siłi sumy momentów). Jest to metoda empiryczna, pozwalająca wyznaczyć bezpieczną (niższą niż uzyskana w badaniach) wartość siły krytycznej. Ta metoda nie opisuje mechanizmu powstawania formy wygiętej pręta. Teoria modułu stycznego nie uwzględnia w całości mechanizmu odkształcenia pręta, tzn. powstawania strefy odciążanej po wypukłej stronie pręta. Teoria uwzględniająca odciążanie po wypukłej stronie pręta zgodnie z charakterystyką modułu sprężystości wzdłużnej E została przedstawiona przez Th. von Karman'a; zwana jest teorią modułu zredukowanego E_r. Zależność na wartość modułu zredukowanego E_r i wartość siły krytycznej (odpowiednio długość wyboczeniową) została wyprowadzona w oparciu o warunek równowagi siłi warunek równowagi momentów działających na wycinek pręta. Pomimo, że teoria modułu zredukowanego E_r uwzględnia odciążanie i ma solidne podstawy teoretyczne, to wartości krytyczne siły wyznaczone na jej podstawie są większe niż uzyskane z badań; czyli metoda nie jest wystarczająco bezpieczna. Ze względu na te rozbieżności, w 1946 r., w Lockheed Aircraft Corporation przeprowadzono test, który miałna celu dokładne sprawdzenie odkształceń pręta (po stronie wklęsłej i po stronie wypukłej) podczas obciążania go powyżej granicy plastyczności. Po badaniach F.R. Shanley opublikowałdwa artykuły. W pierwszym przedstawiłniespójności teorii modułu stycznego E_t i modułu zredukowanego E_r w postaci paradoksu. W drugim artykule, w pierwszej części, przedstawił(w postaci wykresów) dokładne wyniki pomiarów odkształceń na obu bokach pręta i odpowiadające im wartości siły ściskającej. W drugiej części zaproponowałmodel pręta i mechanizm odkształcenia oraz analizę matematyczną. Jednak model pręta bardzo odbiega zarówno od modelu rozpatrywanego w teorii modułu zredukowanego E_r, jak i od pręta rzeczywistego. Koncepcja F.R. Shanley'a była tematem bardzo licznych opracowań teoretycznych dotyczących wyboczenia prętów i płyt. Wymienione wcześniej teorie bazowały jedynie na charakterystyce z próby jednoosiowego rozciągania. Spójna teoria wyboczenia powinna uwzględniać podstawowe zasady wytrzymałości materiałów i być zgodna z wynikami badań. Wyboczenie prętów w zakresie sprężysto-plastycznym było jednym z pierwszych zagadnień wytrzymałości materiałów dotyczącym plastyczności, jednak w opisie mechanizmu wyboczenia nie uwzględniano wytężenia materiału. Jak wynika z teorii modułu zredukowanego E_r, po obciążeniu pręta siłą, która w całym przekroju poprzecznym wywołuje naprężenia równe granicy plastyczności może dojść do redystrybucji naprężeń i zmiany stanu naprężeń z jednorodnego na niejednorodny. W artykule przedstawiono koncepcję mechanizmu wyboczenia prętów, w której uwzględniono wpływ wytężenia materiału według hipotezy Hubera-Misesa-Hencky'ego (HMH). Do opisu mechanizmu zaprezentowano graficzną analizę w oparciu o idealizowaną (dwu-modułową) charakterystykę z próby jednoosiowego rozciągania oraz wyidealizowane, zmienione (zmniejszone) charakterystyki, uwzględniające wpływ naprężeń tnących na wartość naprężeń normalnych pozostających w strefie sprężystej zgodnie z hipotezą HMH. Z przedstawionej analizy koncepcyjnej wynika, że: (1) – w pręcie obciążonym powyżej granicy plastyczności (w rzeczywistym pręcie już powyżej granicy proporcjonalności), następuje samoczynna, tzn. bez zwiększania obciążenia zewnętrznego, redystrybucja naprężeń i odkształceń (tu została nazwana wewnętrzną utratą stateczności). (2) – od tego momentu wystąpienia wewnętrznej utraty stateczności, podczas zwiększania obciążenia

zewnętrznego, nie jest możliwe wywołanie w pręcie jednorodnego stanu naprężenia i odkształcenia. (3) – naprężenia i odkształcenia mogą rozwijać się prawdopodobnie zgodnie z wartością modułu stycznego E_t jednak przy wartościach naprężeń mniejszych niż określone z charakterystyki próby jednoosiowego rozciągania. (4) – rozwój naprężeń i odkształceń zgodnie z charakterystyką z próby jednoosiowego rozciągania jest zbyt dużym uproszczeniem. (5) – dla rzeczywistych charakterystyk z próby jednoosiowego rozciągania (nie wyidealizowanych) wielkości odkształceń pręta będą inne. W celu dokładniejszego zbadania wpływu wytężenia materiału na wartości krytyczne prętów podlegających wyboczeniu w zakresie sprężysto-plastycznym potrzebne są dalsze badania.

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