WARSAW UNIVERSITY OF TECHNOLOGY Index 351733
FACULTY OF CIVIL ENGINEERING
COMMITTEE FOR CIVIL AND WATER ENGINEERING
POLISH ACADEMY OF SCIENCES ISSN 1230-2945

DOI: 10.24425/ace.2025.153349

ARCHIVES OF CIVIL ENGINEERING

Vol. LXXI ISSUE 1 2025

pp. 543 –558

Research paper

Steady-state vibrations of a beam structure under moving horizontal force-analytical approach

Magdalena Ataman[1], Wacław Szcześniak[2]

Abstract: The following paper presents a dynamic analysis of a railway superstructure subjected to horizontal moving loads. The superstructure including the substructure was modelled with two infinitely long beams connected by an elastic layer. The structure rests on a Winkler elastic bed carrying loads in the horizontal direction. The analysed system is loaded by a moving horizontal force that moves at a constant speed tangentially to the upper beam. Similar solution and calculations can be provided for the loaded lower beam. The problem was solved analytically by bringing the two equations of motion into a single differential equation with an higher order. The solution is illustrated with a computational example and the results are analysed in detail. The relationships between the horizontal displacements and the characteristics of the structure and the subsoil, and between the strains and the characteristics of the structure and the subsoil, were shown in the corresponding diagrams obtained using the analytical method. The effect of load velocity on the horizontal displacements and strains of the system under study was also investigated and the results are shown in the figures. The problem solved and the results obtained can be applied to the dynamic analysis of the railway superstructure and the substructure. The results can be used as a benchmark for FEM analysis of more complex engineering structures under moving loads caused by railroad vehicles. The issue is particularly relevant for high-speed railways.

Keywords: analytical solution, beam structure, elastic foundation, horizontal moving load, longitudinal vibration, wave problem

[1]DSc., PhD., Eng. Warsaw University of Technology, Faculty of Civil Engineering, Al. Armii Ludowej 16, 00-637 Warsaw, Poland, e-mail: magdalena.ataman@pw.edu.pl, ORCID: 0000-0002-8048-9840

[2]Prof., DSc., PhD., Eng., Warsaw University of Technology, Faculty of Civil Engineering, Al. Armii Ludowej 16, 00-637 Warsaw, Poland, e-mail: waclaw.szczesniak@pw.edu.pl, ORCID: 0000-0002-5395-2705

1. Introduction

The development of high-speed rail requires detailed dynamic analyses not only of the vehicles but also of the railway superstructure and substructure, including analysis of the vibrations of the structure in both the vertical and horizontal directions. In the literature, rail is usually analysed as a single beam resting on a variety of elastic ground models, often with non-linear elements, which is generally loaded by vertical forces. The first paper on the modelling of railway track under moving loads is the work of Winkler [1], which appeared in 1867. Modelling the railway rail as an infinitely long continuous beam resting on non-deformable hinged supports and loaded by an infinite sequence of forces, Winkler determined the maximum dynamic moment taking into account the centrifugal force resulting from the curvature of the track. Another work on the modelling of railway track is Zimmermann's monograph [2] from 1888. The railway track scheme proposed by Zimmermann is a beam resting on elastic supports (sleepers) and loaded by a moving force. A generalisation of the Zimmermann model to beams with different numbers of supports was presented in 1915 by Timoshenko [3], who is also the author of later works on railway superstructure, e.g. [4]. Other researchers such as Ludwig [5], Dörr [6, 7], Frýba [8], Inglis [9], Bogacz and Popp [10], Bajer and Bogacz [11], Kerr [12, 13] and many others also dealt with the railway track issue. Dynamic phenomena in the vehicle-track system were discussed by Bogacz and Czyczuła in their paper [14]. Based on the results of experimental and theoretical studies, they pointed out that they are particularly relevant for railway lines intended for high-speed train traffic. The influence of the type of railway surface on its vibrations at different train velocities was discussed in the work of Błaszkiewicz and Czyczuła [15]. Issues related to the dynamics of the rail vehicle – track system were also presented by Kisilowski in his monograph [16]. A wide range of issues related to railway superstructure was discussed in detail by Esveld in monograph [17]. A rich review of the literature on this subject can be found in Szcześniak's review paper [18].

There are also works in the literature in which the railway superstructure is modelled with more complex systems, such as two beams connected by an elastic or viscoelastic layer. A series of works by Oniszczuk [19–21], a work by Zbiciak et al. [22], as well as works by Szcześniak and Ataman [23, 24] are devoted, among others, to layered systems, with elastic infill, under vertical force loading.

In the case of moving loads such as trains, both vertical and horizontal forces act on the superstructure. The phenomenon of the occurrence of forces tangential to the circumference of the vehicle wheel has an impact on the dynamic response of the track superstructure and substructure. The authors of relatively few publications have dealt with the description and analysis of this issue. Some of the relevant literature is devoted to methods for estimation and calculation of tangential forces between railway wheel and rail, e.g. [25, 26]. Longitudinal vibrations of beams under horizontal moving loads were analysed, for example, in publications by Nguyen and Duhamel [27], Czyczuła and Chudyba [28] and Ataman [29], as well as Szcześniak and Ataman [30, 31].

The problem of longitudinal vibrations in beams, the so-called wave problem, is already a classical problem in structural dynamics, e.g. monographs by Graff [32], Achenbach [33], Kaliski [34]. Analytical solutions of the wave problem are obtained using Fourier transforms, Laplace transforms and Fourier series. Computer methods, e.g. the finite element method of Ekevid and Wiberg [35], Ekevid et al. [36], Kouroussis et al. [37], are also used to analyse the vibrations of the surface together with the substructure and the phenomenon of wave propagation in these media caused by train movements.

In publications [27–31] in which the authors analysed the effects of moving horizontal forces on the structure, these acted on a single beam resting on a deformable foundation. This paper presents a dynamic analysis of a system of two infinitely long beams with elastic filling. The structure rests on a Winkler foundation acting in the horizontal direction. The structure under study is loaded by a moving horizontal force that moves with constant velocity. The problem is solved using an analytical method and the results are illustrated by graphs obtained at different values of the coefficients describing the properties of the structure and the foundation and at different loading speeds. The analytical solution to the problem under consideration is particularly important as the results obtained can be used as a benchmark for calculating more complex engineering structures under moving loads caused by railroad vehicles. Analytical methods are also used to verify and validate results obtained by approximate methods such as FEM, FDM and others.

2. Governing equations and solution of the problem

We consider a system of two infinitely long beams with elastic filling with the coefficient $k_1$resting on a Winkler foundation described by the coefficient k_2. According to the notations adopted in Fig. 1, the two homogeneous wave equations of motion of the beam system without taking damping into account are written as follows:

$$
\begin{aligned}
&- E_1A_1\frac{\partial^2 u_1}{\partial x^2} + m_1\frac{\partial^2 u_1}{\partial t^2} + k_1(u_1 - u_2) = 0 \\
&- E_2A_2\frac{\partial^2 u_2}{\partial x^2} + m_2\frac{\partial^2 u_2}{\partial t^2} + k_1(u_2 - u_1) + k_2u_2 = 0
\end{aligned}
\tag{2.1}
$$

In Fig. 1 and Eq. (2.1), the following designations are used: $u_1 = u_1(x,t)$, $u_2 = u_2(x,t)$ – horizontal displacements of the top and bottom beams respectively, E_1A_1, E_2A_2 – stiffness

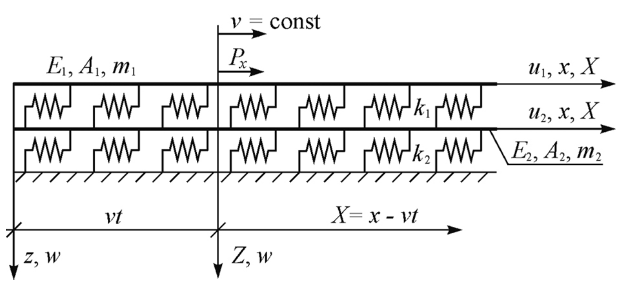

Fig. 1. Dynamic diagram of two infinitely long beams elastically connected, on a Winkler foundation in the horizontal direction, with a horizontal force moving at a constant velocity v

of beams, m_1, m_2 – unit mass densities of the upper and lower beams, with $m_1 = \rho_1 A_1$, $m_2 = \rho_2 A_2$, k_1, k_2 – Winkler coefficients of infill between beams and the foundation on which the beam system rests.

From Eq. (2.1), it is possible to derive differential relationships describing the displacements of the upper beam in relation to the displacements of the lower beam and vice versa:

$$\tag{2.2} \begin{aligned} u_1 = u_1(x,t) &= \frac{1}{k_1}\left[-E_2A_2\frac{\partial^2 u_2}{\partial x^2} + m_2\frac{\partial^2 u_2}{\partial t^2} + u_2(k_1+k_2)\right] \\ u_2 = u_2(x,t) &= \frac{1}{k_1}\left(-E_1A_1\frac{\partial^2 u_1}{\partial x^2} + m_1\frac{\partial^2 u_1}{\partial t^2} + k_1u_1\right) \end{aligned}$$

The second-order homogeneous equations (2.1) and (2.2) can be replaced by a single fourth-order equation per coordinate $u_1 = u_1(x,t)$ or per coordinate $u_2 = u_2(x,t)$. We then have the equations respectively:

$$\tag{2.3} \begin{aligned} &\frac{E_1A_1E_2A_2}{k_1}\frac{\partial^4 u_1}{\partial x^4} + \frac{m_1m_2}{k_1}\frac{\partial^4 u_1}{\partial t^4} - \frac{1}{k_1}(m_2E_1A_1 - m_1E_2A_2)\frac{\partial^4 u_1}{\partial x^2\partial t^2} \\ &\quad + \left(m_2\frac{k_1+k_2}{k_1} + m_2\right)\frac{\partial^2 u_1}{\partial t^2} - \left(\frac{k_1+k_2}{k_1}E_1A_1 + E_2A_2\right)\frac{\partial^2 u_1}{\partial x^2} - k_1u_1 = 0 \\ &\frac{E_1A_1E_2A_2}{k_1}\frac{\partial^4 u_2}{\partial x^4} + \frac{m_1m_2}{k_1}\frac{\partial^4 u_2}{\partial t^4} - \frac{1}{k_1}(m_1E_2A_2 + m_2E_1A_1)\frac{\partial^4 u_2}{\partial x^2\partial t^2} \\ &\quad + \left[m_1\frac{k_2}{k_1} + (m_1+m_2)\right]\frac{\partial^2 u_2}{\partial t^2} - \frac{1}{k_1}\left(\frac{k_1+k_2}{k_1}E_1A_1 + E_2A_2\right)\frac{\partial^2 u_1}{\partial x^2} - k_2u_2 = 0 \end{aligned}$$

The two Eqs. (2.3) are higher order equations with unknowns $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$. Each of these homogeneous equations is of fourth order for both the spatial function and time. Their solution requires four boundary conditions and four initial conditions.

A moving load in the form of a horizontal force, or a continuous load distributed over a certain distance, can occur either on the top or bottom beam or simultaneously on both beams. The load velocity v is assumed constant. In the paper, we will limit the consideration only to the case in which the moving horizontal force $P_x = P$ travels along the upper beam. The load can be expressed in terms of Dirac delta as follows:

$$\tag{2.4} q(x,t) = P\delta(x - vt)$$

The problem is quasi-stationary, so we introduce a moving coordinate system XZ, associated with a moving force:

$$\tag{2.5} \begin{aligned} &X = x - vt, \quad \frac{\partial^2 u_1}{\partial t^2} = v^2\frac{\partial^2 u_1}{\partial X^2}, \quad \frac{\partial^4 u_1}{\partial t^4} = v^4\frac{\partial^4 u_1}{\partial X^4}, \quad \frac{\partial^4 u_1}{\partial X^2\partial t^2} = v^2\frac{\partial^4 u_1}{\partial X^4} \\ &\frac{\partial u_1}{\partial x} = \frac{\partial u_1}{\partial X}, \quad \frac{\partial u_2}{\partial x} = \frac{\partial u_2}{\partial X} \end{aligned}$$

In such a coordinate system, the partial equations of motion (2.1) with the load on the upper beam are of the form:

$$\begin{aligned} & -E_1A_1\frac{\partial^2 u_1}{\partial X^2}+m_1v^2\frac{\partial^2 u_1}{\partial X^2}+k_1(u_1-u_2)=P\delta(X) \\ & -E_2A_2\frac{\partial^2 u_2}{\partial X^2}+m_2v^2\frac{\partial^2 u_2}{\partial X^2}+k_1(u_2-u_1)+k_2u_2=0 \end{aligned} \tag{2.6}$$

Equations (2.6) can be separated into two equations with unknowns $u_1 = u_1(X)$ and $u_2 = u_2(X)$. To do this, we determine the deflection of the lower beam from the first equation of Eqs. (2.6):

$$u_2 = u_2(X) = \frac{1}{k_1}\left[-E_1A_1\frac{\partial^2 u_1}{\partial X^2}+m_1v^2\frac{\partial^2 u_1}{\partial X^2}+k_1(u_1-u_2)-P\delta(X)\right] \tag{2.7}$$

Then substituting u_2 from Eq. (2.7) into the first of Eqs. (2.6), we obtain a fourth-order inhomogeneous equation due to the unknown $u_1 = u_1(X)$:

$$\begin{aligned} & \frac{1}{k_1}[E_1A_1E_2A_2+v^4m_1\,m_2-v^2(m_2E_1A_1+m_1E_2A_2)]\frac{\partial^4 u_1(X)}{\partial X^4} \\ & +\frac{1}{k_1}\{v^4[m_1k_2+(m_1+m_2)k_1]-[k_1E_2A_2+E_1A_1(k_1+k_2)]\}\frac{\partial^2 u_1(X)}{\partial X^2} \\ & +k_2u_1(X)=P\delta(X) \end{aligned} \tag{2.8}$$

where $\delta(x - vt) = \delta(X)$.

Introducing the two longitudinal wave velocities $c_1 = \sqrt{E_1/\rho_1}$ and $c_2 = \sqrt{E_2/\rho_2}$, the homogeneous equation (2.8) is written as follows:

$$\widetilde{A}\frac{\partial^4 u_1(X)}{\partial X^4}+\widetilde{B}\frac{\partial^2 u_1(X)}{\partial X^2}+k_2u_1(X)=0 \tag{2.9}$$

where the coefficients $\widetilde{A}$ and $\widetilde{B}$ are expressed by the formulae:

$$\begin{aligned} \widetilde{A} &= \frac{m_1m_21}{k_1}[v^4-v^2(c_1^2-c_2^2)+c_1^2c_2^2] \\ \widetilde{B} &= \frac{1}{k_1}\{[m_1k_2+(m_1+m_2)k_1]v^2-[k_1E_2A_2+E_1A_1(k_1+k_2)]\} \end{aligned} \tag{2.10}$$

The equation of vibrational motion of the beam (2.9) is a fourth-order equation because the two-beam structure under consideration is a bi-modal system. We solve the equations of vibration of the top and bottom beams using the initial conditions and boundary conditions of the problem. The initial conditions in the top and bottom beams are of the form, respectively:

$$\begin{aligned} & u_1(X,0)=0 \qquad \frac{\partial u_1(X,0)}{\partial t}=0 \\ & u_2(X,0)=0 \qquad \frac{\partial u_2(X,0)}{\partial t}=0 \end{aligned} \tag{2.11}$$

In turn, the boundary and continuity conditions for upper beam are as follows:

$$u_1^a(\infty,t)=0,\qquad u_1^b(-\infty,t)=0,\qquad \left[\frac{\partial u_1^a}{\partial X}-\frac{\partial u_1^b}{\partial X}\right]_{X=0}=-\frac{P}{m(c^2-v^2)}$$
$$\left[\frac{\partial^2 u_1^a}{\partial X^2}-\frac{\partial^2 u_1^b}{\partial X^2}\right]_{X=0}=0,\qquad \left[\frac{\partial^3 u_1^a}{\partial X^3}-\frac{\partial^3 u_1^b}{\partial X^3}\right]_{X=0}=0 \tag{2.12}$$

If there is no load moving on the lower beam, all boundary and continuity conditions, analogous to (2.12), for this beam are zero.

In further considerations we will adopt some simplifications, namely in Eqs. (2.8) and (2.9) we will assume that the material properties and cross-section of the two beams are the same: $E_1=E_2=E$, $\rho_1=\rho_2=\rho$, $A_1=A_2=A$, $m_1=m_2=m$, $k_1=k_2=k$. We will also assume that the velocity of the moving load is less than the longitudinal wave velocity c. With these assumptions, Eq. (2.8) and the coefficients $\widetilde{A}$ and $\widetilde{B}$ simplify considerably and have the following form:

$$\widetilde{A}\frac{\partial^4 u_1(X)}{\partial X^4}+\widetilde{B}\frac{\partial^2 u_1(X)}{\partial X^2}+k_2u_1(X)=0$$
$$\widetilde{A}=\frac{m^2}{k}(v^2-c^2)^2,\qquad \widetilde{B}=3m(v^2-c^2) \tag{2.13}$$

The solution of the homogeneous equation of motion Eq. (2.13) or Eq. (2.9) can be represented in exponential form or as hyperbolic functions. Thus, we have:

$$u_1(X)=Ae^{rx}\ \longrightarrow\ \frac{m^2}{k}(v^2-c^2)^2r^4-3m(v^2-c^2)r^2+k=0 \tag{2.14}$$

After solving the biquadratic Eq. (2.14), we obtain four roots:

$$r_{1/2}=\pm\sqrt{\frac{2k}{(c^2-v^2)(3+\sqrt{5})}},\qquad r_{3/4}=\pm\sqrt{\frac{2k}{(c^2-v^2)(3-\sqrt{5})}} \tag{2.15}$$

The dimension of the elements of r_i is [1/m], and as can easily be seen $r_1=-r_2$ and $r_3=-r_4$. The solution of Eq. (2.13) takes the form:

$$u_1(X,t)=A_1e^{r_1X}+A_2e^{r_2X}+A_3e^{r_3X}+A_4e^{r_4X} \tag{2.16}$$

Due to the boundary conditions, at $X>0$ we must have a vanishing length solution on the right side of the beam, in front of the horizontal force. Thus we have:

$$u_1^a(X)=A_2e^{r_2X}+A_4e^{r_4X}\qquad \text{for } X>0,$$
$$u_1^b(X)=A_1e^{r_1X}+A_3e^{r_3X}\qquad \text{for } X<0 \tag{2.17}$$

Using the conditions (2.12), we obtain the following values for the coefficients A_1–A_4:

$$A_1=A_2=\frac{Pr_4^2}{2mr_2(r_2^2-r_4^2)(c^2-v^2)},\qquad A_3=A_4=\frac{Pr_2^2}{2mr_4(r_4^2-r_2^2)(c^2-v^2)} \tag{2.18}$$

The resulting formulae can be programmed and the results presented graphically. The results of the calculations will be shown in a calculation example.

3. Simplified problem model

For a structure modelled with a single beam, as in Fig. 2, the problem is described by a single differential equation. In the coordinate system xz, associated with the beam, this equation is of the form (3.1):

$$-\frac{\partial}{\partial x}\left[EA\frac{\partial u(x,t)}{\partial x}\right]+m\frac{\partial^2 u(x,t)}{\partial t^2}+k_x u(x,t)=P_x\delta(x-vt) \tag{3.1}$$

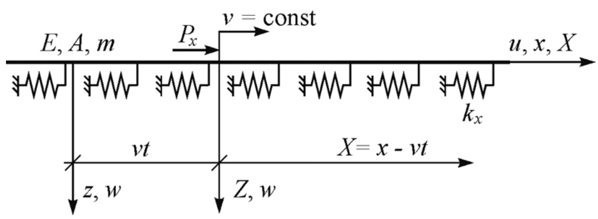

Fig. 2. Infinitely long beam loaded by a moving horizontal force

In the coordinate system XZ, associated with a moving force, the longitudinal vibrations of the beam are described by a second-order ordinary differential equation in which there are no explicit time-dependent members (3.2)

$$-\frac{\partial}{\partial X}\left[EA\frac{\partial u(X)}{\partial X}\right]+\rho A v^2\frac{\partial^2 u(X)}{\partial X^2}+k_x u(X)=P_x\delta(X) \tag{3.2}$$

We solve the problem with boundary conditions in an infinitely long beam

$$u_b(X)|_{X\to-\infty}=0,\qquad u_a(X)|_{X\to\infty}=0 \tag{3.3}$$

and the conditions of continuity of longitudinal displacement and axial force in a section with a coordinate $X=0$

$$u_b(0)=u_a(0),\qquad \rho A(c^2-v^2)\frac{\partial u_a(X)}{\partial X}\bigg|_{X=0}-\rho A(c^2-v^2)\frac{\partial u_b(X)}{\partial X}\bigg|_{X=0}=-P_x \tag{3.4}$$

where $c=\sqrt{E/\rho}$ is the propagation velocity of the longitudinal wave in the beam.

The final solution of the longitudinal vibration problem of an infinitely long beam on an elastic foundation in the coordinate system associated with the moving force is written in two intervals:

– in front of the force

$$u_a(X)=\frac{P_x}{2\sqrt{k_x\rho A(c^2-v^2)}}\,e^{-\sqrt{\frac{k_x}{\rho A(c^2-v^2)}}X}\qquad \text{for } X\geq 0 \tag{3.5}$$

– behind the force

$$u_b(X)=\frac{P_x}{2\sqrt{k_x\rho A(c^2-v^2)}}\,e^{\sqrt{\frac{k_x}{\rho A(c^2-v^2)}}X}\qquad \text{for } X\leq 0 \tag{3.6}$$

In a beam-related system, on the other hand, the horizontal displacement of any point (section) of the beam is expressed by the formulae:

$$u_a(x,t) = \frac{P_x}{2\sqrt{k_x \rho A(c^2 - v^2)}}\, e^{-\sqrt{\frac{k_x}{\rho A(c^2-v^2)}}(x-vt)} \qquad \text{for } x \geq vt \tag{3.7}$$

and

$$u_b(x,vt) = \frac{P_x}{2\sqrt{k_x \rho A(c^2 - v^2)}}\, e^{\sqrt{\frac{k_x}{\rho A(c^2-v^2)}}(x-vt)} \qquad \text{for } x \leq vt \tag{3.8}$$

The deformations of the beam caused by the moving horizontal force in front of and behind the point of application are equal respectively:

$$\begin{aligned} \frac{\partial u_a(x,t)}{\partial x} &= -\frac{P_x}{2k_x \rho A(c^2 - v^2)}\, e^{-\sqrt{\frac{k_x}{\rho A(c^2-v^2)}}(x-vt)} \qquad \text{for } x \geq vt \\ \frac{\partial u_b(x,t)}{\partial x} &= \frac{P_x}{2k_x \rho A(c^2 - v^2)}\, e^{\sqrt{\frac{k_x}{\rho A(c^2-v^2)}}(x-vt)} \qquad \text{for } x \leq vt \end{aligned} \tag{3.9}$$

As can be seen from the solutions obtained, formulae (3.5)–(3.9), and from the expression $\rho A(c^2 - v^2)$, three cases are possible:

1) $v < c$ – the force travels at a speed lower than the velocity of propagation of the longitudinal wave in the beam (subcritical speed),

2) $v = c$ – the force travels at the velocity of propagation of the longitudinal wave in the beam (critical speed),

3) $v > c$ – the force travels at a speed greater than the velocity of propagation of the longitudinal wave in the beam (supercritical speed).

In engineering problems, the first case is of practical importance. Therefore, the computational examples presented in the next chapter are limited to subcritical velocities.

4. Calculation examples

The solutions obtained were programmed onto a computer in Wolfram code ⟨Mathematica 13⟩ and the results presented in graphs. The calculations were carried out for different values of the quantities that characterise the railway superstructure. In order to illustrate the influence of the stiffness of the structure on its displacements and strains, a track on concrete sleepers and on wooden sleepers were analysed.

The examples shown below are for a railway superstructure with the following data: Young's modulus and density of the rail material $E = 200 \cdot 10^9$ Pa and $\rho = 7850\,\text{kg/m}^3$, mass and elasticity coefficient of the foundation in the case of superstructures on concrete sleepers $m = 215.75\,\text{kg/m}$ and $k_x = 19.62$ MPa, in the case of superstructures on wooden sleepers $m = 127.49\,\text{kg/m}$, $k_x = 9.81$ MPa. The horizontal force is $P_x = 15$ kN.

Figures 3–6 show horizontal displacements and strains in a double beam.

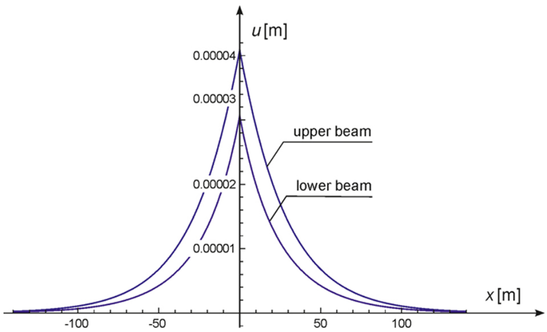

Fig. 3. Horizontal displacements in an infinite double beam on an elastic foundation (concrete sleepers) caused by a horizontal force, $v = 0.01c$

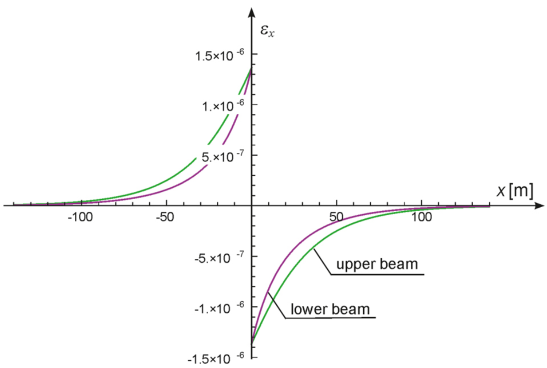

Fig. 4. Horizontal strains in an infinite double beam on an elastic foundation (concrete sleepers) caused by a horizontal force, $v = 0.01c$

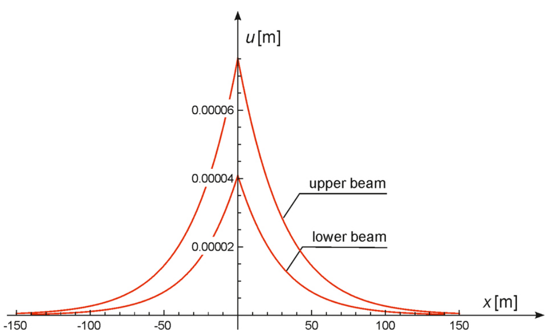

Fig. 5. Horizontal displacements in an infinite double beam on an elastic foundation (wooden sleepers) caused by a horizontal force, $v = 0.01c$

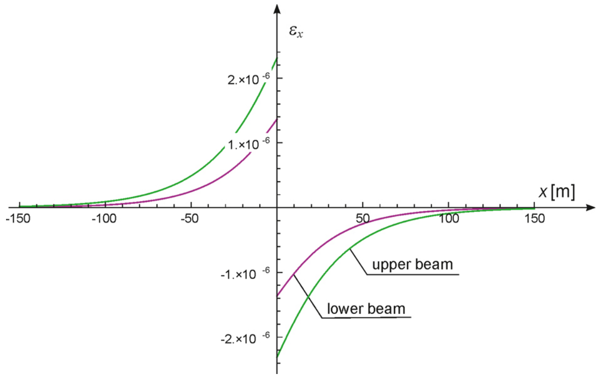

Fig. 6. Horizontal strains in an infinite double beam on an elastic foundation (wooden sleepers) caused by a horizontal force, $v = 0.01c$

Figure 7 presents displacements in a top beam in the case of superstructures on concrete and wooden sleepers.

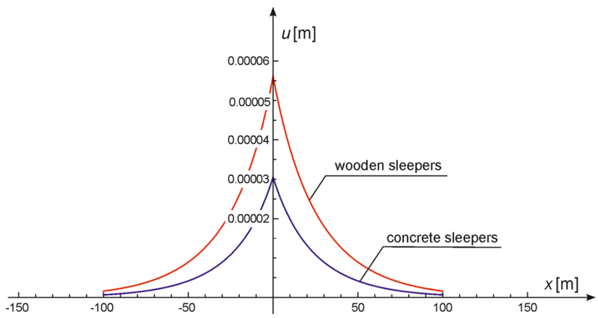

Fig. 7. Horizontal displacements in a top beam caused by a horizontal force, $v = 0.1c$

The graphs presented in Figs. 8 and 9 concern a single beam. In the dynamic case shown in Fig. 8 and Fig. 9, the force moves with a constant velocity equal to half the speed of longitudinal wave propagation in the rail ($v = 0.5c$). This velocity is considerably higher than that of real rail vehicles travelling on existing engineering structures, and the diagrams are for illustrative purposes only.

The use of analytical methods for solving problem allows the influence of individual parameters to be analysed over the full range of speeds and data characterizing rail track and ground. It can be seen (Fig. 10 and Fig. 11) that up to a value of about $v = 0.3c$, the horizontal displacements in the analysed example are practically no different from the values determined in the static case ($v = 0$).

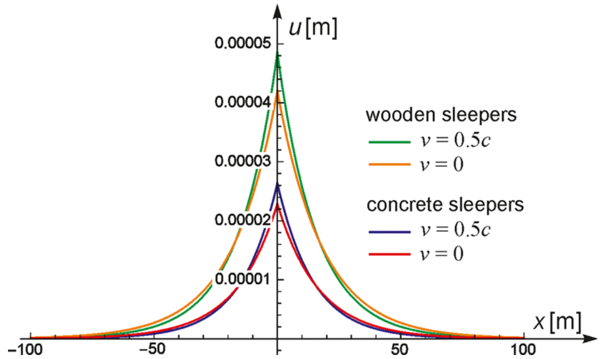

Fig. 8. Horizontal displacements in an infinite single beam on an elastic foundation caused by a horizontal force

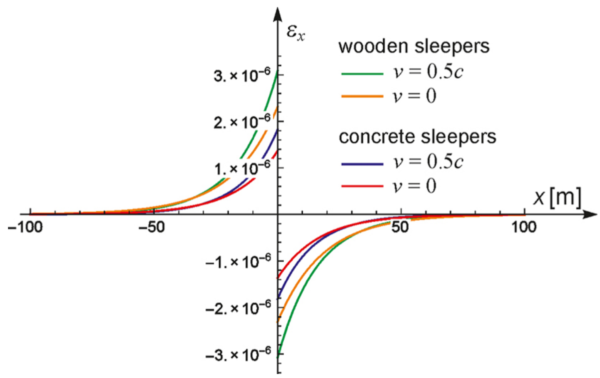

Fig. 9. Horizontal strains in an infinite single beam on an elastic foundation induced by a horizontal force

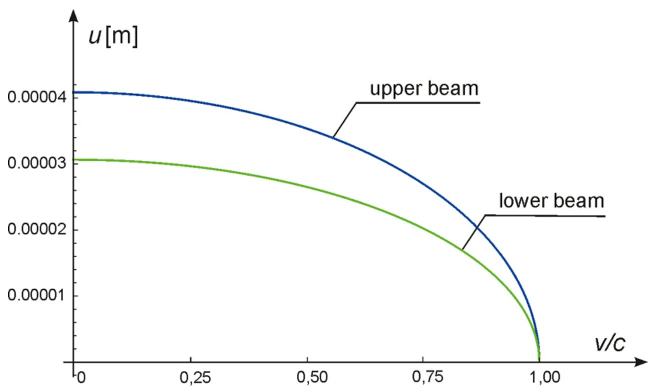

Fig. 10. Horizontal displacements in an infinite double beam on an elastic foundation (concrete sleepers) as a function of v/c

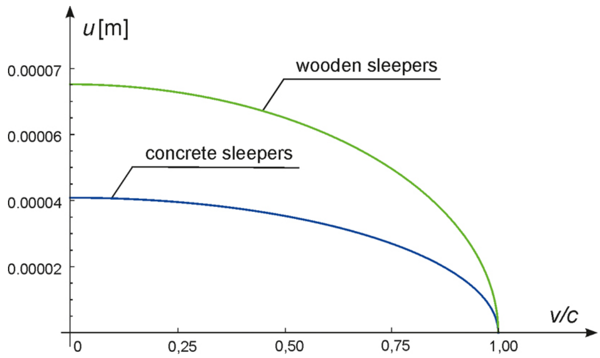

Fig. 11. Horizontal displacements in a top beam caused by a horizontal force, as a function of v/c

5. Conclusions

On the basis of the calculations carried out, it can be concluded that the horizontal displacements above the velocity $v = 0.3c$ are practically no different from the values determined in the static case ($v = 0$). In the adopted model, the velocity of the longitudinal wave is the speed of propagation of sound in steel. In contrast, the actual velocity of the longitudinal wave in the railway superstructure is much lower than the velocity in the steel rail ($c = \sqrt{E/\rho} \approx 5047\,\text{m/s}$).

The proposed railway superstructure models are applicable to the dynamic analysis of both ballasted and ballastless superstructures.

The use of analytical methods for solving problems allows the influence of individual parameters to be analysed over the full range of speeds and data characterizing superstructure and ground.

The solutions presented in this paper can be applied to the study of the stability of a railway track both in plan and in profile, taking into account the relevant beam stiffness and parameters describing the soil substrate. The analytical solutions obtained can be used to verify the results obtained by the Finite Element Method.

The longitudinal wave phenomenon is also important in the design of bearings and abutments in railway and road bridges. Longitudinal forces are also important when designing modern hyperloop transport [38]. They should also be taken into account in the dynamic analysis of curved railway tracks [39].

In the longer perspective, the effect of the forces caused by the snaking of rail vehicles on the dynamic response of the track superstructure and substructure should be further investigated. This phenomenon may be of particular relevance in the case of high-speed train traffic.

Acknowledgements

This paper was co-financed under the research grant of the Warsaw University of Technology supporting the scientific activity in the discipline of Civil Engineering and Transport.

References

[1] E. Winkler, *Die Lehre von der Elastizität und Festigkeit (The theory of elasticity and strength).* Prag: Dominicus, 1867 (in German).

[2] H. Zimmermann, *Die Berechnung des Eisenbahn-Oberbaues (The calculation of the railway superstructure).* Berlin: Ernst & Korn, 1888 (in German).

[3] С.П. Тимошенко, К вопросу о прочности рельс (On the issue of rail strength). Petrograd: A.E. Collins, 1915 (in Russian).

[4] S.P. Timoshenko, "Method of analysis of statical and dynamical stresses in rail", in *Proceedings of the Second International Congress for Applied Mechanics*. Zurich, Switzerlan, 1926, pp. 407–418.

[5] K. Ludwig, "*Die Verformung eines beiderseits unbegrenzten elastisch gebetteten Geleises durch Lasten mit konstanter Horizontalgeschwindigkeit" (Deformation of a rail elastically supported and infinite length by loads moving at a constant horizontal velocity)*, in Proceedings of the Fifth International Congress for Applied Mechanics, Cambridge, Massachusetts, 1938, pp. 650–655.

[6] J. Dörr, "Das Schwingungsverhalten eines federnd gebetteten unendlich langen Balkens" (The vibration behavior of a resiliently supported infinitely long beam), *Ingenieur-Archiv*, vol. 16, pp. 287–298, 1948, doi: 10.1007/bf00534494 (in German).

[7] J. Dörr, "Der unendliche, federnd gebettete Balken unter dem Einflu\ einer gleichförmig bewegten Last" (The infinite, resiliently supported beam under the influence of a uniformly moving load), *Ingenieur-Archiv*, vol. 14, pp. 167–192, 1943, doi: 10.1007/BF02085990.

[8] L. Frýba, *Dynamics of railway Bridges*. London: Thomas Telford, 1996.

[9] C.E. Inglis, *A mathematical treatise on vibrations in railway bridges.* London: Cambridge University Press, 1934.

[10] R. Bogacz and K. Popp, *"Dynamics and stability of train-track-systems"*, in *Proceedings of 2nd International Conference on Recent Advances in Structural Dynamics*. Southampton, England: 1984, vol. 2, pp. 709–721.

[11] C.I. Bajer and R. Bogacz, "Propagation of perturbances generated in classic track, and track with Y-type sleepers", *Archives of Applied Mechanics*, vol. 74, pp. 754–761, 2005, doi: 10.1007/s00419-005-0403-8.

[12] A.D. Kerr, "On the stability of the railroad track in the vertical plan", *Rail International*, no. 2, pp. 132–142, 1974.

[13] A.D. Kerr, "The stress and stability analysis of railroad tracks", *Journal of Applied Mechanics*, vol. 41, no. 4, pp. 841–848, 1974, doi: 10.1115/1.3423470.

[14] R. Bogacz and W. Czyczuła, "Wybrane zagadnienia dynamiki kolei dużych prędkości, przegląd badań dotyczących toru" (Selected problems of high-speed railway dynamics, review of track research), *Zeszyty Naukowo-Techniczne SITK RP, Oddziałw Krakowie (Scientific and Technical Journals of SITK RP, Branch in Cracow)*, no. 2(101), pp. 59–72, 2013 (in Polish).

[15] D.K. Błaszkiewicz and W. Czyczuła, "Ocena wpływu konstrukcji nawierzchni na parametry jej drgań przy różnych prędkościach pociągu" (Assessment of the impact of the railroad structure on its parameters of vibration at various train speeds),

Zeszyty Naukowo-Techniczne SITK RP, Oddziałw Krakowie (Scientific and Technical Journals of SITK RP, Branch in Cracow), no. 2(101), pp. 47–57, 2013 (in Polish).

[16] J. Kisilowski, Ed., *Dynamika układu mechanicznego pojazd szynowy – tor (Dynamics of the mechanical system: rail vehicle – track).* Warsaw, Poland: PWN Scientific Publishing House, 1991.

[17] C. Esveld, *Modern Railway Track*. Delft University of Technology: MRT-Productions, 1989, 2002.

[18] W. Szcześniak, *Wybrane zagadnienia kolejowe (Selected railway problems).* Warsaw University of Technology Research Papers, Construction no. 129. Warsaw, Poland: Publishing House of the Warsaw University of Technology, 1995 (in Polish).

[19] Z. Oniszczuk, "Drgania poprzeczne układu dwóch belek połączonych elementem sprężystym" (Transversal vibration of the system of two beams connected by means of an elastic element), *Journal of Theoretical and Applied Mechanics*, vol. 12, no. 1, pp. 72–83, 1974 (in Polish).

[20] Z. Oniszczuk, "Transverse Vibrations of Elastically Connected Double-String Complex System. Part I: Free Vibrations", *Journal of Sound and Vibration*, vol. 232, no. 2, pp. 355–366, 2000, doi: 10.1006/jsvi.1999.2742.

[21] Z. Oniszczuk, "Transverse Vibrations of Elastically Connected Double-String Complex System. Part II: Forced Vibrations", *Journal of Sound and Vibration*, vol. 232, no. 2, pp. 367–386, 2000, doi: 10.1006/jsvi.1999.2743.

[22] A. Zbiciak, M. Ataman, and W. Szcześniak, "Use of the finite element method in predicting vibrations of sandwich beams and plates resting on deformable foundations subjected to moving loads", *Archives of Civil Engineering*, vol. 63, no. 4, pp. 51–69, 2017, doi: 10.1515/ace-2017-0040.

[23] W. Szcześniak and M. Ataman, "Układ dwóch nieskończenie długich belek z wypełnieniem sprężystym pod obciążeniem ruchomym" (A system of two infinite beams separated by an elastic core under moving load), *Efektywność Transportu, Autobusy*, vol. 19, no. 12, pp. 285–288, 2018, doi: 10.24136/atest.2018.400 (in Polish).

[24] M. Ataman and W. Szcześniak, "Vibrations of the plate with elastic core resting on the Winkler foundation caused by a moving oscillator", *Transportation Overview – Przegląd Komunikacyjny*, no. 11, pp. 29–38, 2019, doi: 10.35117/A_ENG_19_11_04.

[25] T. Sano, Y. Endo, S. Nakazawa, and D. Hijikata, "On Estimation of Tangential Force in Railways Brake Systems by Fuzzy Inference", *Journal Journal of Advanced Computational Intelligence and Intelligent Informatics*, vol. 19 no. 5, pp. 639–644, 2015, doi: 10.20965/jaciii.2015.p0639.

[26] N. Bosso and N. Zampieri, "A Novel Analytical Method to Calculate Wheel-Rail Tangential Forces and Validation on a Scaled Roller-Rig",

Advances in Tribology, vol. 2018, art. no. 7298236, pp. 1–11, 2018, doi: 10.1155/2018/7298236.

[27] V-H. Nguyen and D. Duhamel, "Finite element procedures for nonlinear structures in moving coordinates. Part 1: Infinite bar under moving axial loads", *Computers and Structures*, vol. 84, no. 21, pp. 1368–1380, 2006, doi:10.1016/j.compstruc.2006.02.018.

[28] W. Czyczuła and Ł. Chudyba, "Dynamic response of rail track in longitudinal direction", *Transportation Overview*, no. 7, pp. 2–6, 2018, doi: 10.35117/a_eng_18_07_01.

[29] M. Ataman, *Analiza drgań nawierzchni i podtorza pod wpływem obciążeń ruchomych z dużymi prędkościami (Dynamic analysis of a railway superstructure and subgrade subjected to high-speed moving loads)*. Warsaw, Poland: Publishing House of the Warsaw University of Technology, 2019 (in Polish).

[30] W. Szcześniak and M. Ataman, "Drgania podłużne belki mostowej pod wpływem poziomego, ruchomej siły poziomej" (Longitudinal vibrations of the bridge beam forced by the moving horizontal force), *TTS Technika Transportu Szynowego*, no. 12, pp. 2959–2963, 2015 (in Polish).

[31] W. Szcześniak and M. Ataman, "Drgania podłużne belki mostowej pod wpływem poziomego, ruchomego obciążenia inercyjnego" (Longitudinal vibrations of the bridge beam forced by the moving inertial load), *TTS Technika Transportu Szynowego*, no. 12, pp. 2955–2956, 2015 (in Polish).

[32] K.F. Graff, *Wave motion in elastic solids*. Elsevier, 1999.

[33] J.D. Achenbach, *Wave propagation in elastic solids*. Elsevier, 1976.

[34] S. Kaliski, *Drgania i fale (Vibrations and waves)*. Warsaw, Poland: PWN Scientific Publishing House, 1966 (in Polish).

[35] T. Ekevid and N-E. Wiberg, "Wave propagation related to high-speed train: a scaled boundary FE-approach for unbounded domains", *Computer Methods in Applied Mechanics and Engineering*, vol. 191, no. 36, pp. 3947–3964, 2002, doi: 10.1016/S0045-7825(02)00341-9.

[36] T. Ekevid, M.X.D. Li, and N-E. Wiberg, "Adaptive FEA of wave propagation induced by high-speed trains", *Computers and Structures*, vol. 79, no. 29-30, pp. 2693–2704, 2001, doi: 10.1016/S0045-7949(01)00043-8.

[37] G. Kouroussis, D.P. Connolly, and O. Verlinden, "Railway-induced ground vibrations – a review of vehicle effects", *International Journal of Rail Transportation*, vol. 2, no. 2, pp. 69–110, 2014, doi:10.1080/23248378.2014.897791.

[38] H. Zobel, A. Pawlak, M. Pawlik, P. Żółtowski, R. Czubacki, and Th. Al-Khafaji, "Hyperloop- Civil engineering point of view according to Polish experience", *Archives of Civil Engineering*, vol. 68, no. 1, pp. 1–25, 2022, doi: 10.24425/ace.2022.140153.

[39] K. Zboiński and P. Woźnica, "Railway transition curves – optimization and assessment", *Archives of Civil Engineering*, vol. 68, no. 1, pp. 207–221, 2022, doi: 10.24425/ace.2022.140164.

Drgania stacjonarne układu belek z wypełnieniem sprężystym na podłożu odkształcalnym pod ruchomą siłą poziomą – podejście analityczne

Słowa kluczowe: drgania podłużne, podłoże sprężyste, poziome obciążenie ruchome, rozwiązanie analityczne, układ belek z wypełnieniem sprężystym, zagadnienie falowe

Streszczenie:

Rozwój kolei dużych prędkości wymaga szczegółowych analiz dynamicznych nie tylko pojazdów, ale również nawierzchni i podtorza kolejowego, w tym analizy drgań konstrukcji zarówno w kierunku pionowym jak i w kierunku poziomym. W literaturze przedmiotu szynę kolejową analizuje się zwykle jako pojedynczą belkę spoczywającą na rozmaitych modelach podłoża sprężystego, nierzadko z elementami nieliniowymi, która obciążona jest na ogółsiłami pionowymi. Pierwszym opracowaniem dotyczącym modelowania toru kolejowego pod obciążeniem ruchomym jest praca Winklera [1], która ukazała się w 1867 roku. Modelując szynę kolejową jako nieskończenie długą belkę ciągłą spoczywającą na nieodkształcalnych podporach przegubowych i obciążoną nieskończonym ciągiem siłskupionych, Winkler wyznaczyłmaksymalny moment dynamiczny z uwzględnieniem siły odśrodkowej wynikającej z krzywizny toru. Kolejną pracą dotyczącą modelowania toru kolejowego jest monografia Zimmermanna [2] z 1888 roku. Schemat nawierzchni kolejowej zaproponowany przez Zimmermanna jest belką spoczywającą na podporach sprężystych (podkładach) i obciążoną ruchomą siłą skupioną. Uogólnienie modelu Zimmermanna na belki o różnej liczbie podpór przedstawiłw 1915 roku Timoshenko [3], który jest także autorem późniejszych prac na temat nawierzchni kolejowej, np. [4]. Zagadnieniem toru kolejowego zajmowali się także inni badacze, tacy jak: Ludwig [5], Dörr [6, 7], Frýba [8], Inglis [9], Bogacz i Popp [10], Bajer i Bogacz [11], Kerr [12, 13] oraz wielu innych. Zjawiska dynamiczne w układzie pojazd–tor omówili Bogacz i Czyczuła w pracy [14]. Bazując na wynikach badań eksperymentalnych i teoretycznych, wskazali, że są one szczególnie istotne w przypadku linii kolejowych przeznaczonych do ruchu pociągów dużych prędkości. Wpływ rodzaju nawierzchni kolejowej na jej drgania przy różnych prędkościach pociągu zostałomówiony w pracy Błaszkiewicz i Czyczuły [15]. Zagadnienia związane z dynamiką układu pojazd szynowy – tor przedstawiłrównież Kisilowski w monografii [16]. Szeroki zakres problematyki dotyczącej nawierzchni kolejowych omówiłszczegółowo Esveld w monografii [17]. Bogaty przegląd literatury dotyczącej tej tematyki znaleźć można w opracowaniu przeglądowym Szcześniaka [18]. W literaturze istnieją również prace, w których nawierzchnia kolejowa modelowana jest bardziej złożonymi układami, np. dwóch belek połączonych warstwą sprężystą lub lepkosprężystą. Układom warstwowym, z wypełnieniem sprężystym, pod obciążeniem siłami pionowymi poświęcony jest między innymi cykl prac Oniszczuka [19–21], praca Zbiciaka i innych [22], a także prace Szcześniaka i Ataman [23, 24]. W przypadku obciążeń ruchomych jakimi są pociągi na konstrukcję nawierzchni oddziałują zarówno siły pionowe jak i poziome. Zjawisko występowania siłstycznych do obwodu koła pojazdu ma wpływ na odpowiedź dynamiczną nawierzchni kolejowej i podtorza. Opisem i analizą tego zagadnienia zajmowali się autorzy stosunkowo niewielu publikacji. Drgania podłużne belek pod wpływem poziomych obciążeń ruchomych analizowane były na przykład w publikacjach Nguyen i Duhamel [27], Czyczuła i Chudyba [28] oraz Ataman [29], a także Szcześniak i Ataman [30, 31]. Zagadnienie drgań podłużnych w belkach, tzw. zadanie falowe, jest już zadaniem klasycznym w dynamice konstrukcji, np. monografie Graff [32], Achenbach [33], Kaliski [34]. Rozwiązania analityczne zadania falowego uzyskuje się z wykorzystaniem transformacji Fouriera, transformacji Laplace'a oraz szeregów Fouriera. Do analizy drgań nawierzchni wraz z podtorzem oraz zjawiska rozchodzenia się fal w tych ośrodkach, wywołanych ruchem pociągów, wykorzystuje się również metody komputerowe, np. metodę elementów skończonych Ekevid i Wiberg [35],

Ekevid i inni [36], Kouroussis i inni [37]. W publikacjach [29–33], w których autorzy analizowali wpływ ruchomych siłpoziomych na konstrukcję, działały one na pojedynczą belkę spoczywającą na podłożu odkształcalnym. W artykule przedstawiono analizę dynamiczną nawierzchni kolejowej poddanej działaniu poziomych obciążeń ruchomych. Konstrukcja nawierzchni wraz z podtorzem zamodelowana została dwiema nieskończenie długimi belkami połączonymi warstwą sprężystą. Struktura spoczywa na podłożu sprężystym Winklera przenoszącym obciążenia w kierunku poziomym. Analizowany układ obciążony jest ruchomą siłą poziomą, która przesuwa się ze stałą prędkością stycznie do górnej lub dolnej belki. Zadanie rozwiązano analitycznie doprowadzając dwa równania ruchu do jednego równania różniczkowego o zawyżonym rzędzie. Zaproponowane modele nawierzchni kolejowej mają zastosowanie do analizy dynamicznej zarówno nawierzchni podsypkowych, jak i bezpodsypkowych. Zastosowanie analitycznych metod rozwiązania zadań pozwala na analizę wpływu poszczególnych parametrów w pełnym zakresie prędkości oraz danych charakteryzujących nawierzchnię i podłoże. Przedstawione w pracy rozwiązania mogą mieć zastosowanie do badania stateczności toru kolejowego zarówno w planie jak i w profilu, przy uwzględnieniu odpowiednich sztywności belki oraz parametrów opisujących podłoże gruntowe. Uzyskane rozwiązania analityczne mogą być wykorzystane do weryfikacji wyników otrzymanych metodą elementów skończonych. Zjawisko fali podłużnej jest także istotne przy projektowaniu łożysk i przyczółków w mostach kolejowych i drogowych. W dalszej perspektywie należałoby dodatkowo przebadać wpływ siłwywołanych wężykowaniem pojazdów szynowych na odpowiedź dynamiczną nawierzchni i podtorza. Przeanalizowane w artykule zagadnienie jest szczególnie istotne w przypadku kolei dużych prędkości.

Received: 2023-12-01, Revised: 2024-02-21